Problem 29

In each of Problems 29 through 32, use the method outlined in Problem 28 to solve the given differential equation.

$$t^2y'' - 2ty' + 2y = 4t^2, \quad t > 0; \qquad y_1(t) = t$$

Solution

Since one solution is known, the method of reduction of order can be applied to determine the general solution. Substitute $y(t) = c(t)t$ into the ODE and solve the resulting ODE for $c(t)$.

$$t^2[c(t)t]'' - 2t[c(t)t]' + 2[c(t)t] = 4t^2$$

Evaluate the derivatives.

$$t^2[c'(t)t + c(t)]' - 2t[c'(t)t + c(t)] + 2[c(t)t] = 4t^2$$
$$t^2[c''(t)t + c'(t) + c'(t)] - 2t[c'(t)t + c(t)] + 2[c(t)t] = 4t^2$$
$$t^3c''(t) + \cancel{2t^2c'(t)} - \cancel{2t^2c'(t)} - \cancel{2tc(t)} + \cancel{2tc(t)} = 4t^2$$
$$t^3c''(t) = 4t^2$$

Divide both sides by t^3.

$$c''(t) = \frac{4}{t}$$

Integrate both sides with respect to t.

$$c'(t) = 4\ln t + C_1$$

Integrate both sides with respect to t once more.

$$\begin{aligned} c(t) &= \int^t 4\ln s\, ds + C_1t + C_2 \\ &= 4\int^t \frac{d}{ds}(s)\ln s\, ds + C_1t + C_2 \\ &= 4\left[(s)\ln s\bigg|^t - \int^t (s)\left(\frac{1}{s}\right) ds\right] + C_1t + C_2 \\ &= 4\left(t\ln t - \int^t ds\right) + C_1t + C_2 \\ &= 4(t\ln t - t) + C_1t + C_2 \end{aligned}$$

Therefore, using a new constant C_3 for $-4 + C_1$,

$$\begin{aligned} y(t) &= c(t)t \\ &= [4(t\ln t - t) + C_1t + C_2]t \\ &= 4t^2\ln t - 4t^2 + C_1t^2 + C_2t \\ &= 4t^2\ln t + C_3t^2 + C_2t. \end{aligned}$$

The terms containing C_3 and C_2 are the second and first solutions, respectively, to the associated homogeneous equation, and the first term is the particular solution.