Problem 1

In each of Problems 1 through 4, use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

$$y'' - 5y' + 6y = 2e^t$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 5y_c' + 6y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 5(re^{rt}) + 6(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

$$r = \{2, 3\}$$

Two solutions to equation (1) are then $y_c = e^{2t}$ and $y_c = e^{3t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1 e^{2t} + C_2 e^{3t}$$

Method of Variation of Parameters

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^{2t} + C_2(t)e^{3t}$$

Substitute this into the original ODE to determine $C_1(t)$ and $C_2(t)$.

$$[C_1(t)e^{2t} + C_2(t)e^{3t}]'' - 5[C_1(t)e^{2t} + C_2(t)e^{3t}]' + 6[C_1(t)e^{2t} + C_2(t)e^{3t}] = 2e^t$$

Evaluate the derivatives.

$$[C_1'(t)e^{2t} + 2C_1(t)e^{2t} + C_2'(t)e^{3t} + 3C_2(t)e^{3t}]' - 5[C_1'(t)e^{2t} + 2C_1(t)e^{2t} + C_2'(t)e^{3t} + 3C_2(t)e^{3t}] + 6[C_1(t)e^{2t} + C_2(t)e^{3t}] = 2e^t$$

$$[C_1''(t)e^{2t} + 2C_1'(t)e^{2t} + 2C_1'(t)e^{2t} + 4C_1(t)e^{2t} + C_2''(t)e^{3t} + 3C_2'(t)e^{3t} + 3C_2'(t)e^{3t} + 9C_2(t)e^{3t}] \\ - 5[C_1'(t)e^{2t} + 2C_1(t)e^{2t} + C_2'(t)e^{3t} + 3C_2(t)e^{3t}] + 6[C_1(t)e^{2t} + C_2(t)e^{3t}] = 2e^t$$

Simplify the left side.

$$[C_1''(t) - C_1'(t)]e^{2t} + [C_2''(t) + C_2'(t)]e^{3t} = 2e^t$$

If we set

$$C_2''(t) + C_2'(t) = 0, \tag{2}$$

then the previous equation reduces to

$$[C_1''(t) - C_1'(t)]e^{2t} = 2e^t. \tag{3}$$

The aim now is to solve this system of equations. Solve equation (2) first for $C_2(t)$. Integrate both sides with respect to t, setting the integration constant to zero.

$$C_2'(t) + C_2(t) = 0$$

Use an integrating factor I_1 to solve this ODE.

$$I_1 = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of the previous equation by I_1.

$$e^t C_2'(t) + e^t C_2(t) = 0$$

The left side can be written as $d/dt[I_1 C_2(t)]$ by the product rule.

$$\frac{d}{dt}[e^t C_2(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^t C_2(t) = 0$$

Divide both sides by e^t.

$$C_2(t) = 0$$

Now solve equation (3) for $C_1(t)$. Start by dividing both sides of it by e^{2t}.

$$C_1''(t) - C_1'(t) = 2e^{-t}$$

Use an integrating factor I_2 to solve this ODE.

$$I_2 = \exp\left[\int^t (-1)\, ds\right] = e^{-t}$$

Multiply both sides of the previous equation by I_2.

$$e^{-t}C_1''(t) - e^{-t}C_1'(t) = 2e^{-2t}$$

The left side can be written as $d/dt[I_2 C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[e^{-t}C_1'(t)] = 2e^{-2t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^{-t}C_1'(t) = -e^{-2t}$$

Multiply both sides by e^t.

$$C_1'(t) = -e^{-t}$$

Integrate both sides with respect to t once more.

$$C_1(t) = e^{-t}$$

Consequently, the particular solution is

$$\begin{aligned} y_p(t) &= C_1(t)e^{2t} + C_2(t)e^{3t} \\ &= e^t. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1e^{2t} + C_2e^{3t} + e^t. \end{aligned}$$

Method of Undetermined Coefficients

The particular solution satisfies

$$y_p'' - 5y_p' + 6y_p = 2e^t.$$

Since the inhomogeneous term is an exponential function, use $y_p(t) = Ae^t$ for the trial solution. Substitute this into the ODE to determine A.

$$(Ae^t)'' - 5(Ae^t)' + 6(Ae^t) = 2e^t$$

Evaluate the derivatives.

$$(Ae^t) - 5(Ae^t) + 6(Ae^t) = 2e^t$$

$$2Ae^t = 2e^t$$

For this equation to be true, A must satisfy $2A = 2$, which means $A = 1$. That means the particular solution is

$$y_p(t) = e^t.$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1e^{2t} + C_2e^{3t} + e^t. \end{aligned}$$