Problem 13

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

$$t^2y'' - 2y = 3t^2 - 1, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^{-1}$$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$t^2y_1'' - 2y_1 \stackrel{?}{=} 0$$

$$t^2(t^2)'' - 2(t^2) \stackrel{?}{=} 0$$

$$t^2(2) - 2(t^2) \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$t^2y_2'' - 2y_2 \stackrel{?}{=} 0$$

$$t^2\left(\frac{1}{t}\right)'' - 2\left(\frac{1}{t}\right) \stackrel{?}{=} 0$$

$$t^2\left(\frac{2}{t^3}\right) - 2\left(\frac{1}{t}\right) \stackrel{?}{=} 0$$

$$\frac{2}{t} - \frac{2}{t} \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

By the principle of superposition, $y_c(t)$ is a linear combination of $y_1(t)$ and $y_2(t)$.

$$y_c(t) = C_1t^2 + C_2t^{-1}$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)t^2 + C_2(t)t^{-1}$$

It satisfies the following ODE.

$$t^2y_p'' - 2y_p = 3t^2 - 1$$

Substitute the previous formula for $y_p(t)$.

$$t^2[C_1(t)t^2 + C_2(t)t^{-1}]'' - 2[C_1(t)t^2 + C_2(t)t^{-1}] = 3t^2 - 1$$

Evaluate the derivatives.

$$t^2[C_1'(t)t^2 + 2C_1(t)t + C_2'(t)t^{-1} - C_2(t)t^{-2}]' - 2[C_1(t)t^2 + C_2(t)t^{-1}] = 3t^2 - 1$$

$$\begin{aligned} t^2[C_1''(t)t^2 + 2C_1'(t)t + 2C_1'(t)t + 2C_1(t) + C_2''(t)t^{-1} - C_2'(t)t^{-2} - C_2'(t)t^{-2} + 2C_2(t)t^{-3}] \\ - 2[C_1(t)t^2 + C_2(t)t^{-1}] = 3t^2 - 1 \end{aligned}$$

$$\begin{aligned} C_1''(t)t^4 + 2C_1'(t)t^3 + 2C_1'(t)t^3 + \cancel{2C_1(t)t^2} + C_2''(t)t - C_2'(t) - C_2'(t) + \cancel{2C_2(t)t^{-1}} \\ - \cancel{2C_1(t)t^2} - \cancel{2C_2(t)t^{-1}} = 3t^2 - 1 \end{aligned}$$

$$C_1''(t)t^4 + 4C_1'(t)t^3 + C_2''(t)t - 2C_2'(t) = 3t^2 - 1$$

If we set

$$C_2''(t)t - 2C_2'(t) = 0, \tag{1}$$

then the previous equation reduces to

$$C_1''(t)t^4 + 4C_1'(t)t^3 = 3t^2 - 1. \tag{2}$$

The aim now is to solve this system of equations for $C_1(t)$ and $C_2(t)$. Divide both sides of equation (1) by t.

$$C_2''(t) - \frac{2}{t}C_2'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t -\frac{2}{s}\,ds\right) = e^{-2\ln t} = e^{\ln t^{-2}} = t^{-2}$$

Multiply both sides of the previous equation by I_1.

$$\frac{1}{t^2}C_2''(t) - \frac{2}{t^3}C_2'(t) = 0$$

The left side can be written as $d/dt[I_1 C_2'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\frac{1}{t^2}C_2'(t)\right] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$\frac{1}{t^2}C_2'(t) = 0$$

Multiply both sides by t^2.

$$C_2'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_2(t) = 0$$

Divide both sides of equation (2) by t^4.

$$C_1''(t) + \frac{4}{t}C_1'(t) = \frac{3}{t^2} - \frac{1}{t^4}$$

Use an integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t \frac{4}{s}\,ds\right) = e^{4\ln t} = e^{\ln t^4} = t^4$$

Multiply both sides of the previous equation by I_2.

$$t^4 C_1''(t) + 4t^3 C_1'(t) = 3t^2 - 1$$

The left side can be written as $d/dt[I_2 C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[t^4 C_1'(t)] = 3t^2 - 1$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$t^4 C_1'(t) = t^3 - t$$

Divide both sides by t^4.

$$C_1'(t) = \frac{1}{t} - \frac{1}{t^3}$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_1(t) = \ln t + \frac{1}{2t^2}$$

The particular solution is then

$$\begin{aligned} y_p(t) &= C_1(t)y_1(t) + C_2(t)y_2(t) \\ &= C_1(t)t^2 + C_2(t)t^{-1} \\ &= \left(\ln t + \frac{1}{2t^2}\right)t^2 \\ &= t^2 \ln t + \frac{1}{2}. \end{aligned}$$

Therefore, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 t^2 + C_2 t^{-1} + t^2 \ln t + \frac{1}{2}. \end{aligned}$$