Problem 14

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

$$t^2y'' - t(t+2)y' + (t+2)y = 2t^3, \quad t > 0; \qquad y_1(t) = t, \quad y_2(t) = te^t$$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$t^2y_1'' - t(t+2)y_1' + (t+2)y_1 \stackrel{?}{=} 0$$

$$t^2(t)'' - t(t+2)(t)' + (t+2)(t) \stackrel{?}{=} 0$$

$$-t(t+2)(1) + (t+2)(t) \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$t^2y_2'' - t(t+2)y_2' + (t+2)y_2 \stackrel{?}{=} 0$$

$$t^2(te^t)'' - t(t+2)(te^t)' + (t+2)(te^t) \stackrel{?}{=} 0$$

$$t^2(e^t + te^t)' - t(t+2)(e^t + te^t) + (t+2)(te^t) \stackrel{?}{=} 0$$

$$t^2(e^t + e^t + te^t) - te^t(t+2) - t^2e^t(t+2) + te^t(t+2) \stackrel{?}{=} 0$$

$$2t^2e^t + t^3e^t - te^t(t+2) - t^3e^t - 2t^2e^t + te^t(t+2) \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

By the principle of superposition, $y_c(t)$ is a linear combination of $y_1(t)$ and $y_2(t)$.

$$y_c(t) = C_1t + C_2te^t$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)t + C_2(t)te^t$$

It satisfies the following ODE.

$$t^2y_p'' - t(t+2)y_p' + (t+2)y_p = 2t^3$$

Substitute the previous formula for $y_p(t)$.

$$t^2[C_1(t)t + C_2(t)te^t]'' - t(t+2)[C_1(t)t + C_2(t)te^t]' + (t+2)[C_1(t)t + C_2(t)te^t] = 2t^3$$

Evaluate the derivatives.

$$t^2[C_1'(t)t+C_1(t)+C_2'(t)te^t+C_2(t)e^t+C_2(t)te^t]'-t(t+2)[C_1'(t)t+C_1(t)+C_2'(t)te^t+C_2(t)e^t+C_2(t)te^t] \\ + (t+2)[C_1(t)t + C_2(t)te^t] = 2t^3$$

$$t^2[C_1''(t)t+C_1'(t)+C_1'(t)+C_2''(t)te^t+C_2'(t)e^t+C_2'(t)te^t+C_2'(t)e^t+C_2(t)e^t+C_2'(t)te^t+C_2(t)e^t+C_2(t)te^t] \\ - t(t+2)[C_1'(t)t + C_1(t) + C_2'(t)te^t + C_2(t)e^t + C_2(t)te^t] \\ + (t+2)[C_1(t)t + C_2(t)te^t] = 2t^3$$

$$C_1''(t)t^3+2t^2C_1'(t)+C_2''(t)t^3e^t+C_2'(t)t^2e^t+C_2'(t)t^3e^t+C_2'(t)t^2e^t+C_2(t)t^2e^t+C_2'(t)t^3e^t+C_2(t)t^2e^t+C_2(t)t^3e^t \\ -C_1'(t)t^3-C_1(t)t^2-C_2'(t)t^3e^t-C_2(t)t^2e^t-C_2(t)t^3e^t-2C_1'(t)t^2-2C_1(t)t-2C_2'(t)t^2e^t-2C_2(t)te^t-2C_2(t)t^2e^t \\ + C_1(t)t^2 + C_2(t)t^2e^t + 2C_1(t)t + 2C_2(t)te^t = 2t^3$$

$$t^3C_1''(t) - t^3C_1'(t) + t^3e^tC_2''(t) + t^3e^tC_2'(t) = 2t^3$$

Divide both sides by t^3.

$$C_1''(t) - C_1'(t) + e^tC_2''(t) + e^tC_2'(t) = 2$$

If we set

$$C_1''(t) - C_1'(t) = 0, \tag{1}$$

then the previous equation reduces to

$$e^tC_2''(t) + e^tC_2'(t) = 2. \tag{2}$$

The aim now is to solve this system of equations for $C_1(t)$ and $C_2(t)$. Use an integrating factor I_1 to solve equation (1).

$$I_1 = \exp\left(\int^t (-1)\,ds\right) = e^{-t}$$

Multiply both sides of equation (1) by I_1.

$$e^{-t}C_1''(t) - e^{-t}C_1'(t) = 0$$

The left side can be written as $d/dt[I_1C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[e^{-t}C_1'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^{-t}C_1'(t) = 0$$

Multiply both sides by e^t.

$$C_1'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_1(t) = 0$$

Divide both sides of equation (2) by e^t.

$$C_2''(t) + C_2'(t) = 2e^{-t}$$

Use an integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t ds\right) = e^t$$

Multiply both sides of the previous equation by I_2.

$$e^t C_2''(t) + e^t C_2'(t) = 2$$

The left side can be written as $d/dt[I_2 C_2'(t)]$ by the product rule.

$$\frac{d}{dt}[e^t C_2'(t)] = 2$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^t C_2'(t) = 2t$$

Divide both sides by e^t.

$$C_2'(t) = 2te^{-t}$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$\begin{aligned}
C_2(t) &= \int^t 2se^{-s}\,ds \\
&= 2\int^t s\frac{d}{ds}(-e^{-s})\,ds \\
&= 2\left[s(-e^{-s})\Big|^t - \int^t (1)(-e^{-s})\,ds\right] \\
&= 2\left[t(-e^{-t}) + \int^t e^{-s}\,ds\right] \\
&= 2(-te^{-t} - e^{-t}) \\
&= -2e^{-t}(t+1)
\end{aligned}$$

The particular solution is then

$$\begin{aligned}
y_p(t) &= C_1(t)y_1(t) + C_2(t)y_2(t) \\
&= C_1(t)t + C_2(t)te^t \\
&= -2t(t+1) \\
&= -2t^2 - 2t.
\end{aligned}$$

Therefore, the general solution is

$$\begin{aligned}
y(t) &= C_1 t + C_2 te^t - 2t^2 - 2t \\
&= (C_1 - 2)t + C_2 te^t - 2t^2 \\
&= C_3 t + C_2 te^t - 2t^2,
\end{aligned}$$

where a new constant C_3 was used for $C_1 - 2$.