Problem 2

In each of Problems 1 through 4, use the method of variation of parameters to find a particular solution of the given differential equation. Then check your answer by using the method of undetermined coefficients.

$$y'' - y' - 2y = 2e^{-t}$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - y_c' - 2y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} - (re^{rt}) - 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - r - 2 = 0$$

$$(r-2)(r+1) = 0$$

$$r = \{-1, 2\}$$

Two solutions to equation (1) are then $y_c = e^{-t}$ and $y_c = e^{2t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_1e^{-t} + C_2e^{2t}$$

<u>**Method of Variation of Parameters**</u>

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_1(t)e^{-t} + C_2(t)e^{2t}$$

Substitute this into the original ODE to determine $C_1(t)$ and $C_2(t)$.

$$[C_1(t)e^{-t} + C_2(t)e^{2t}]'' - [C_1(t)e^{-t} + C_2(t)e^{2t}]' - 2[C_1(t)e^{-t} + C_2(t)e^{2t}] = 2e^{-t}$$

Evaluate the derivatives.

$$[C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^{2t} + 2C_2(t)e^{2t}]' - [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^{2t} + 2C_2(t)e^{2t}] \\ - 2[C_1(t)e^{-t} + C_2(t)e^{2t}] = 2e^{-t}$$

$$[C_1''(t)e^{-t} - C_1'(t)e^{-t} - C_1'(t)e^{-t} + C_1(t)e^{-t} + C_2''(t)e^{2t} + 2C_2'(t)e^{2t} + 2C_2'(t)e^{2t} + 4C_2(t)e^{2t}] \\ - [C_1'(t)e^{-t} - C_1(t)e^{-t} + C_2'(t)e^{2t} + 2C_2(t)e^{2t}] - 2[C_1(t)e^{-t} + C_2(t)e^{2t}] = 2e^{-t}$$

Simplify the left side.

$$[C_1''(t) - 3C_1'(t)]e^{-t} + [C_2''(t) + 3C_2'(t)]e^{2t} = 2e^{-t}$$

If we set

$$C_2''(t) + 3C_2'(t) = 0, \tag{2}$$

then the previous equation reduces to

$$[C_1''(t) - 3C_1'(t)]e^{-t} = 2e^{-t} \tag{3}$$

The aim now is to solve this system of equations. Solve equation (2) first for $C_2(t)$. Integrate both sides with respect to t, setting the integration constant to zero.

$$C_2'(t) + 3C_2(t) = 0$$

Use an integrating factor I_1 to solve this ODE.

$$I_1 = \exp\left(\int^t 3\,ds\right) = e^{3t}$$

Multiply both sides of the previous equation by I_1.

$$e^{3t}C_2'(t) + 3e^{3t}C_2(t) = 0$$

The left side can be written as $d/dt[I_1C_2(t)]$ by the product rule.

$$\frac{d}{dt}[e^{3t}C_2(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^{3t}C_2(t) = 0$$

Divide both sides by e^{3t}.

$$C_2(t) = 0$$

Now solve equation (3) for $C_1(t)$. Start by multiplying both sides of it by e^t.

$$C_1''(t) - 3C_1'(t) = 2$$

Use an integrating factor I_2 to solve this ODE.

$$I_2 = \exp\left[\int^t (-3)\,ds\right] = e^{-3t}$$

Multiply both sides of the previous equation by I_2.

$$e^{-3t}C_1''(t) - 3e^{-3t}C_1'(t) = 2e^{-3t}$$

The left side can be written as $d/dt[I_2C_1'(t)]$ by the product rule.

$$\frac{d}{dt}[e^{-3t}C_1'(t)] = 2e^{-3t}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^{-3t}C_1'(t) = -\frac{2}{3}e^{-3t}$$

Multiply both sides by e^{3t}.

$$C_1'(t) = -\frac{2}{3}$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_1(t) = -\frac{2}{3}t$$

Consequently, the particular solution is

$$\begin{aligned} y_p(t) &= C_1(t)e^{-t} + C_2(t)e^{2t} \\ &= -\frac{2}{3}te^{-t}. \end{aligned}$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1e^{-t} + C_2e^{2t} - \frac{2}{3}te^{-t}. \end{aligned}$$

Method of Undetermined Coefficients

The particular solution satisfies

$$y_p'' - y_p' - 2y_p = 2e^{-t}.$$

Since the inhomogeneous term is an exponential function, we would use $y_p(t) = Ae^{-t}$ for the trial solution. However, since e^{-t} satisfies equation (1), an extra factor of t is needed. Use $y_p(t) = Ate^{-t}$ for the trial solution. Substitute this into the ODE to determine A.

$$(Ate^{-t})'' - (Ate^{-t})' - 2(Ate^{-t}) = 2e^{-t}.$$

Evaluate the derivatives.

$$(-2Ae^{-t} + Ate^{-t}) - (Ae^{-t} - Ate^{-t}) - 2(Ate^{-t}) = 2e^{-t}$$

$$-3Ae^{-t} = 2e^{-t}$$

For this equation to be true, A must satisfy $-3A = 2$, which means $A = -2/3$. That means the particular solution is

$$y_p(t) = -\frac{2}{3}te^{-t}.$$

Therefore,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1e^{-t} + C_2e^{2t} - \frac{2}{3}te^{-t}. \end{aligned}$$