Problem 20

In each of Problems 13 through 20, verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation. In Problems 19 and 20, g is an arbitrary continuous function.

$$x^2y'' + xy' + (x^2 - 0.25)y = g(x), \quad x > 0; \qquad y_1(x) = x^{-1/2}\sin x, \quad y_2(x) = x^{-1/2}\cos x$$

Solution

Verify that the first solution satisfies the associated homogeneous equation.

$$x^2y_1'' + xy_1' + (x^2 - 0.25)y_1 \stackrel{?}{=} 0$$

$$x^2(x^{-1/2}\sin x)'' + x(x^{-1/2}\sin x)' + (x^2 - 0.25)(x^{-1/2}\sin x) \stackrel{?}{=} 0$$

$$x^2\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right)' + x\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right) + (x^2 - 0.25)(x^{-1/2}\sin x) \stackrel{?}{=} 0$$

$$x^2\left(\frac{3}{4}x^{-5/2}\sin x - \frac{1}{2}x^{-3/2}\cos x - \frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right) + x\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right) + (x^2 - 0.25)(x^{-1/2}\sin x) \stackrel{?}{=} 0$$

$$\cancel{\frac{3}{4}x^{-1/2}}\sin x - \cancel{\frac{1}{2}x^{1/2}\cos x} - \cancel{\frac{1}{2}x^{1/2}\cos x} - \cancel{x^{3/2}\sin x} - \cancel{\frac{1}{2}x^{-1/2}\sin x} + \cancel{x^{1/2}\cos x} + \cancel{x^{3/2}\sin x} - \cancel{0.25x^{-1/2}}\sin x \stackrel{?}{=} 0$$

$$0 = 0$$

Now verify that the second solution satisfies the associated homogeneous equation.

$$x^2y_2'' + xy_2' + (x^2 - 0.25)y_2 \stackrel{?}{=} 0$$

$$x^2(x^{-1/2}\cos x)'' + x(x^{-1/2}\cos x)' + (x^2 - 0.25)(x^{-1/2}\cos x) \stackrel{?}{=} 0$$

$$x^2\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right)' + x\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right) + (x^2 - 0.25)(x^{-1/2}\cos x) \stackrel{?}{=} 0$$

$$x^2\left(\frac{3}{4}x^{-5/2}\cos x + \frac{1}{2}x^{-3/2}\sin x + \frac{1}{2}x^{-3/2}\sin x - x^{-1/2}\cos x\right) + x\left(-\frac{1}{2}x^{-3/2}\cos x - x^{-1/2}\sin x\right) + (x^2 - 0.25)(x^{-1/2}\cos x) \stackrel{?}{=} 0$$

$$\cancel{\frac{3}{4}x^{-1/2}}\cos x + \cancel{\frac{1}{2}x^{1/2}\sin x} + \cancel{\frac{1}{2}x^{1/2}\sin x} - \cancel{x^{3/2}\cos x} - \cancel{\frac{1}{2}x^{-1/2}\cos x} - \cancel{x^{1/2}\sin x} + \cancel{x^{3/2}\cos x} - \cancel{0.25x^{-1/2}}\cos x \stackrel{?}{=} 0$$

$$0 = 0$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(x)$ and the particular solution $y_p(x)$.

$$y(x) = y_c(x) + y_p(x)$$

By the principle of superposition, $y_c(x)$ is a linear combination of $y_1(x)$ and $y_2(x)$.

$$y_c(x) = C_1 x^{-1/2} \sin x + C_2 x^{-1/2} \cos x$$

According to the method of variation of parameters, the particular solution is found by allowing the parameters in $y_c(x)$ to vary.

$$y_p(x) = C_1(x) x^{-1/2} \sin x + C_2(x) x^{-1/2} \cos x$$

It satisfies the following ODE.

$$x^2 y_p'' + x y_p' + (x^2 - 0.25) y_p = g(x)$$

Substitute the previous formula for $y_p(x)$.

$$\begin{aligned} x^2[C_1(x)x^{-1/2}\sin x + C_2(x)x^{-1/2}\cos x]'' + x[C_1(x)x^{-1/2}\sin x + C_2(x)x^{-1/2}\cos x]' \\ + (x^2 - 0.25)[C_1(x)x^{-1/2}\sin x + C_2(x)x^{-1/2}\cos x] = g(x) \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned} x^2 \bigg[C_1'(x)x^{-1/2}\sin x - \frac{1}{2}C_1(x)x^{-3/2}\sin x + C_1(x)x^{-1/2}\cos x + C_2'(x)x^{-1/2}\cos x \\ -\frac{1}{2}C_2(x)x^{-3/2}\cos x - C_2(x)x^{-1/2}\sin x \bigg]' \\ + x \bigg[C_1'(x)x^{-1/2}\sin x - \frac{1}{2}C_1(x)x^{-3/2}\sin x + C_1(x)x^{-1/2}\cos x + C_2'(x)x^{-1/2}\cos x \\ -\frac{1}{2}C_2(x)x^{-3/2}\cos x - C_2(x)x^{-1/2}\sin x \bigg] \\ + (x^2 - 0.25)[C_1(x)x^{-1/2}\sin x + C_2(x)x^{-1/2}\cos x] = g(x) \end{aligned}$$

$$\begin{aligned} x^2 \bigg[C_1''(x)x^{-1/2}\sin x - \frac{1}{2}C_1'(x)x^{-3/2}\sin x + C_1'(x)x^{-1/2}\cos x - \frac{1}{2}C_1'(x)x^{-3/2}\sin x + \frac{3}{4}C_1(x)x^{-5/2}\sin x \\ - \frac{1}{2}C_1(x)x^{-3/2}\cos x + C_1'(x)x^{-1/2}\cos x - \frac{1}{2}C_1(x)x^{-3/2}\cos x - C_1(x)x^{-1/2}\sin x \\ + C_2''(x)x^{-1/2}\cos x - \frac{1}{2}C_2'(x)x^{-3/2}\cos x - C_2'(x)x^{-1/2}\sin x - \frac{1}{2}C_2'(x)x^{-3/2}\cos x \\ +\frac{3}{4}C_2(x)x^{-5/2}\cos x + \frac{1}{2}C_2(x)x^{-3/2}\sin x - C_2'(x)x^{-1/2}\sin x + \frac{1}{2}C_2(x)x^{-3/2}\sin x - C_2(x)x^{-1/2}\cos x \bigg] \\ + x \bigg[C_1'(x)x^{-1/2}\sin x - \frac{1}{2}C_1(x)x^{-3/2}\sin x + C_1(x)x^{-1/2}\cos x + C_2'(x)x^{-1/2}\cos x \\ -\frac{1}{2}C_2(x)x^{-3/2}\cos x - C_2(x)x^{-1/2}\sin x \bigg] \\ + (x^2 - 0.25)[C_1(x)x^{-1/2}\sin x + C_2(x)x^{-1/2}\cos x] = g(x) \end{aligned}$$

Simplify the left side.

$$x^{3/2}(\sin x)C_1''(x) + 2x^{3/2}(\cos x)C_1'(x) + x^{3/2}(\cos x)C_2''(x) - 2x^{3/2}(\sin x)C_2'(x) = g(x)$$

Divide both sides by $x^{3/2}$.

$$(\sin x)C_1''(x) + 2(\cos x)C_1'(x) + (\cos x)C_2''(x) - 2(\sin x)C_2'(x) = g(x)x^{-3/2}$$

If we set

$$(\cos x)C_2''(x) - 2(\sin x)C_2'(x) = 0, \tag{1}$$

then the previous equation reduces to

$$(\sin x)C_1''(x) + 2(\cos x)C_1'(x) = g(x)x^{-3/2}. \tag{2}$$

The aim now is to solve this system of equations for $C_1(x)$ and $C_2(x)$. Divide both sides of equation (1) by $\cos x$.

$$C_2''(x) - 2\frac{\sin x}{\cos x}C_2'(x) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^x -2\frac{\sin s}{\cos s}\,ds\right) = e^{2\ln\cos x} = e^{\ln\cos^2 x} = \cos^2 x$$

Multiply both sides of the previous equation by I_1.

$$(\cos^2 x)C_2''(x) - (2\sin x\cos x)C_2'(x) = 0$$

The left side can be written as $d/dx[I_1C_2'(x)]$ by the product rule.

$$\frac{d}{dx}[(\cos^2 x)C_2'(x)] = 0$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$(\cos^2 x)C_2'(x) = 0$$

Divide both sides by $\cos^2 x$.

$$C_2'(x) = 0$$

Integrate both sides with respect to x once more, setting the integration constant to zero.

$$C_2(x) = 0$$

Divide both sides of equation (2) by $\sin x$.

$$C_1''(x) + 2\frac{\cos x}{\sin x}C_1'(x) = \frac{g(x)x^{-3/2}}{\sin x}$$

Use another integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^x 2\frac{\cos s}{\sin s}\,ds\right) = e^{2\ln\sin x} = e^{\ln\sin^2 x} = \sin^2 x$$

Multiply both sides of the previous equation by I_2.

$$(\sin^2 x)C_1''(x) + (2\cos x \sin x)C_1'(x) = g(x)x^{-3/2}\sin x$$

The left side can be written as $d/dx[I_2C_1'(x)]$ by the product rule.

$$\frac{d}{dx}[(\sin^2 x)C_1'(x)] = g(x)x^{-3/2}\sin x$$

Integrate both sides with respect to x, setting the integration constant to zero.

$$(\sin^2 x)C_1'(x) = \int^x g(s)s^{-3/2}\sin s\, ds$$

Divide both sides by $\sin^2 x$.

$$C_1'(x) = \csc^2 x \int^x g(s)s^{-3/2}\sin s\, ds$$

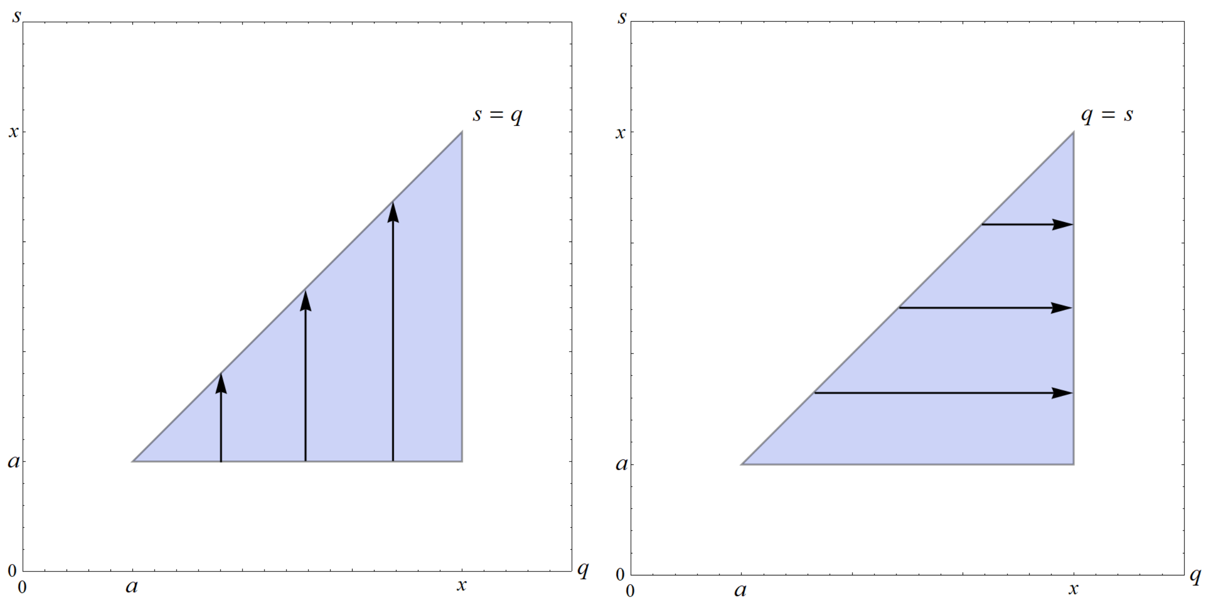
Integrate both sides with respect to x once more, setting the integration constant to zero.

$$\begin{aligned} C_1(x) &= \int^x \csc^2 q \int^q g(s)s^{-3/2}\sin s\, ds\, dq \\ &= \int^x \int^q (\csc^2 q)g(s)s^{-3/2}\sin s\, ds\, dq \end{aligned}$$

Suppose that the initial conditions are both given at $x = a$: $y(a) = y_0$ and $y'(a) = v_0$. Then the lower limits of integration are both a.

$$C_1(x) = \int_a^x \int_a^q (\csc^2 q)g(s)s^{-3/2}\sin s\, ds\, dq$$

The current mode of integration in the qs-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order of integration.

$$\begin{aligned}
C_1(x) &= \int_a^x \int_s^x (\csc^2 q) g(s) s^{-3/2} \sin s \, dq \, ds \\
&= \int_a^x (-\cot q)\bigg|_s^x g(s) s^{-3/2} \sin s \, dq \, ds \\
&= \int_a^x (\cot s - \cot x) g(s) s^{-3/2} \sin s \, ds \\
&= \int_a^x (\cos s - \sin s \cot x) g(s) s^{-3/2} \, ds
\end{aligned}$$

The particular solution is then

$$\begin{aligned}
y_p(x) &= C_1(x) y_1(x) + C_2(x) y_2(x) \\
&= C_1(x) x^{-1/2} \sin x + C_2(x) x^{-1/2} \cos x \\
&= \left[\int_a^x (\cos s - \sin s \cot x) g(s) s^{-3/2} \, ds\right] x^{-1/2} \sin x \\
&= x^{-1/2} \int_a^x (\sin x \cos s - \sin s \cos x) g(s) s^{-3/2} \, ds \\
&= x^{-1/2} \int_a^x \sin(x - s) g(s) s^{-3/2} \, ds.
\end{aligned}$$

Therefore, the general solution is

$$\begin{aligned}
y(x) &= y_c(x) + y_p(x) \\
&= C_1 x^{-1/2} \sin x + C_2 x^{-1/2} \cos x + x^{-1/2} \int_a^x \sin(x - s) g(s) s^{-3/2} \, ds,
\end{aligned}$$

where, again, $x = a$ is when the initial conditions are given.