Problem 24

Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = (D-a)(D-b)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0,$$

where a and b are real numbers with $a \neq b$.

Solution

Method Using Operator Factorization

Since the operator has been factored, the substitution $u = (D-b)y$ is invited. The ODE then becomes

$$(D-a)u = g(t).$$

As a result, the second-order ODE has been reduced to a system of (decoupled) first-order ODEs.

$$\begin{cases} (D-b)y = u & \rightarrow \quad y' - by = u(t) \\ (D-a)u = g & \rightarrow \quad u' - au = g(t) \end{cases}$$

Solve the one for u first by using an integrating factor I_1.

$$I_1 = \exp\left[\int^t (-a)\,ds\right] = e^{-at}$$

Multiply both sides of the ODE for u by I_1.

$$e^{-at}u' - ae^{-at}u = g(t)e^{-at}$$

The left side can be written as $d/dt(I_1 u)$ by the product rule.

$$\frac{d}{dt}(e^{-at}u) = g(t)e^{-at}$$

Integrate both sides with respect to t.

$$e^{-at}u = \int^t g(s)e^{-as}\,ds + C_1$$

Multiply both sides by e^{at}.

$$u(t) = e^{at}\int^t g(s)e^{-as}\,ds + C_1 e^{at}$$

Substitute this result into the ODE for y.

$$y' - by = e^{at}\int^t g(s)e^{-as}\,ds + C_1 e^{at}$$

Use another integrating factor I_2 to solve it.

$$I_2 = \exp\left[\int^t (-b)\,ds\right] = e^{-bt}$$

Multiply both sides of the previous equation by I_2.

$$e^{-bt}y' - be^{-bt}y = e^{at}e^{-bt}\int^t g(s)e^{-as}\,ds + C_1e^{at}e^{-bt}$$

The left side can be written as $d/dt(I_2y)$ by the product rule.

$$\frac{d}{dt}(e^{-bt}y) = e^{(a-b)t}\int^t g(s)e^{-as}\,ds + C_1e^{(a-b)t}$$

Integrate both sides with respect to t.

$$e^{-bt}y = \int^t e^{(a-b)q}\int^q g(s)e^{-as}\,ds\,dq + \frac{C_1}{a-b}e^{(a-b)t} + C_2$$

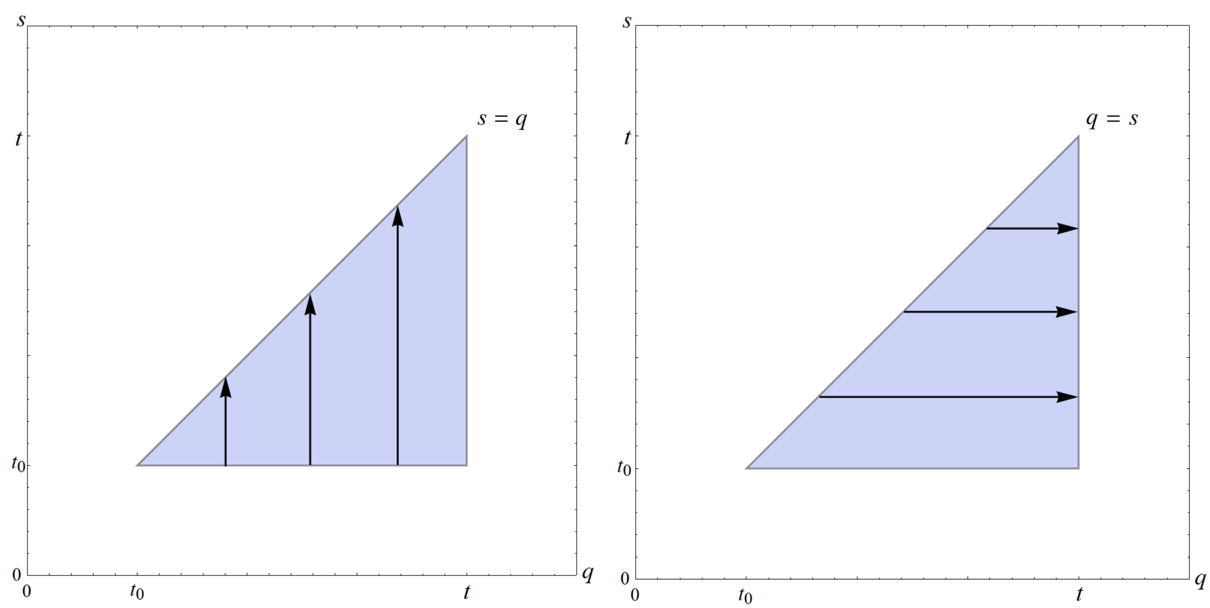
Multiply both sides by e^{bt} and use a new constant C_3 for $C_1/(a-b)$.

$$\begin{aligned} y(t) &= e^{bt}\int^t e^{(a-b)q}\int^q g(s)e^{-as}\,ds\,dq + C_3e^{at} + C_2e^{bt} \\ &= e^{bt}\int^t\int^q e^{(a-b)q}g(s)e^{-as}\,ds\,dq + C_3e^{at} + C_2e^{bt} \end{aligned}$$

Since the initial conditions are given at $t = t_0$, the lower limits of integration will be set to t_0.

$$y(t) = e^{bt}\int_{t_0}^t\int_{t_0}^q e^{(a-b)q}g(s)e^{-as}\,ds\,dq + C_3e^{at} + C_2e^{bt}$$

The current mode of integration in the qs-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order of integration.

$$\begin{aligned} y(t) &= e^{bt}\int_{t_0}^t\int_s^t e^{(a-b)q}g(s)e^{-as}\,dq\,ds + C_3e^{at} + C_2e^{bt} \\ &= e^{bt}\int_{t_0}^t\left(\frac{1}{a-b}e^{(a-b)q}\bigg|_s^t\right)g(s)e^{-as}\,ds + C_3e^{at} + C_2e^{bt} \\ &= \frac{e^{bt}}{a-b}\int_{t_0}^t[e^{(a-b)t} - e^{(a-b)s}]g(s)e^{-as}\,ds + C_3e^{at} + C_2e^{bt} \end{aligned}$$

As a result, the general solution is

$$y(t) = \frac{1}{a-b}\int_{t_0}^{t}[e^{a(t-s)} - e^{b(t-s)}]g(s)\,ds + C_3e^{at} + C_2e^{bt}.$$

Use the Leibniz rule,

$$\frac{d}{dt}\int_{j(t)}^{h(t)} f(t,s)\,ds = \int_{j(t)}^{h(t)} \frac{\partial}{\partial t}f(t,s)\,ds + \frac{dh}{dt}f[t,h(t)] - \frac{dj}{dt}f[t,j(t)],$$

to differentiate the general solution.

$$\begin{aligned} y'(t) &= \frac{1}{a-b}\int_{t_0}^{t}\frac{\partial}{\partial t}[e^{a(t-s)} - e^{b(t-s)}]g(s)\,ds + 1\cdot(e^0 - e^0)g(t) + C_3ae^{at} + C_2be^{bt} \\ &= \frac{1}{a-b}\int_{t_0}^{t}[ae^{a(t-s)} - be^{b(t-s)}]g(s)\,ds + C_3ae^{at} + C_2be^{bt} \end{aligned}$$

Apply the initial conditions now to determine C_2 and C_3.

$$\begin{aligned} y(t_0) &= C_3e^{at_0} + C_2e^{bt_0} = 0 \\ y'(t_0) &= C_3ae^{at_0} + C_2be^{bt_0} = 0 \end{aligned}$$

This system is only satisfied if $C_2 = 0$ and $C_3 = 0$. Therefore,

$$y(t) = \frac{1}{a-b}\int_{t_0}^{t}[e^{a(t-s)} - e^{b(t-s)}]g(s)\,ds.$$

Method Using Variation of Parameters

$$L[y] = (D-a)(D-b)y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0$$

Distribute the operator to obtain a second-order ODE.

$$D(D-b)y - a(D-b)y = g(t)$$

$$y'' - by' - ay' + aby = g(t)$$

$$y'' - (a+b)y' + aby = g(t)$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - (a+b)y_c' + aby_c = 0$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} - (a+b)(re^{rt}) + ab(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - (a+b)r + ab = 0$$

$$r = \frac{(a+b) \pm \sqrt{(a+b)^2 - 4ab}}{2} = \frac{a+b \pm \sqrt{(a-b)^2}}{2} = \frac{a+b \pm (a-b)}{2}$$

$$r = \{a, b\}$$

Two solutions to equation (1) are then $y_c = e^{at}$ and $y_c = e^{bt}$. By the principle of superposition, the general solution is a linear combination of these two.

$$y_c(t) = C_4e^{at} + C_5e^{bt}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_4(t)e^{at} + C_5(t)e^{bt}$$

It satisfies the following ODE.

$$y_p'' - (a+b)y_p' + aby_p = g(t)$$

Substitute the previous formula for $y_p(t)$.

$$[C_4(t)e^{at} + C_5(t)e^{bt}]'' - (a+b)[C_4(t)e^{at} + C_5(t)e^{bt}]' + ab[C_4(t)e^{at} + C_5(t)e^{bt}] = g(t)$$

Evaluate the derivatives.

$$[C_4'(t)e^{at} + aC_4(t)e^{at} + C_5'(t)e^{bt} + bC_5(t)e^{bt}]' \\ - (a+b)[C_4'(t)e^{at} + aC_4(t)e^{at} + C_5'(t)e^{bt} + bC_5(t)e^{bt}] + ab[C_4(t)e^{at} + C_5(t)e^{bt}] = g(t)$$

$$[C_4''(t)e^{at} + aC_4'(t)e^{at} + aC_4'(t)e^{at} + a^2C_4(t)e^{at} + C_5''(t)e^{bt} + bC_5'(t)e^{bt} + bC_5'(t)e^{bt} + b^2C_5(t)e^{bt}] \\ - (a+b)[C_4'(t)e^{at} + aC_4(t)e^{at} + C_5'(t)e^{bt} + bC_5(t)e^{bt}] + ab[C_4(t)e^{at} + C_5(t)e^{bt}] = g(t)$$

Simplify the left side.

$$e^{at}C_4''(t) + ae^{at}C_4'(t) - be^{at}C_4'(t) + e^{bt}C_5''(t) + be^{bt}C_5'(t) - ae^{bt}C_5'(t) = g(t)$$

If we set

$$e^{at}C_4''(t) + ae^{at}C_4'(t) + e^{bt}C_5''(t) + be^{bt}C_5'(t) = 0, \tag{2}$$

then the previous equation reduces to

$$-be^{at}C_4'(t) - ae^{bt}C_5'(t) = g(t). \tag{3}$$

The aim now is to solve this system of two equations for $C_4(t)$ and $C_5(t)$. Start by rewriting equation (2).

$$\frac{d}{dt}[e^{at}C_4'(t)] + \frac{d}{dt}[e^{bt}C_5'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$e^{at}C_4'(t) + e^{bt}C_5'(t) = 0$$

Solve for $C_4'(t)$.

$$C_4'(t) = -\frac{e^{bt}}{e^{at}}C_5'(t) \tag{4}$$

Plug this formula into equation (3).

$$-be^{at}\left[-\frac{e^{bt}}{e^{at}}C_5'(t)\right] - ae^{bt}C_5'(t) = g(t)$$

$$C_5'(t)(be^{bt} - ae^{bt}) = g(t)$$

Divide both sides by $be^{bt} - ae^{bt}$.

$$C_5'(t) = \frac{g(t)e^{-bt}}{b-a}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_5(t) = \int^t \frac{g(s)e^{-bs}}{b-a}\,ds$$

Substitute the previous formula for $C_5'(t)$ into equation (4).

$$\begin{aligned} C_4'(t) &= -\frac{e^{bt}}{e^{at}}\left[\frac{g(t)e^{-bt}}{b-a}\right] \\ &= -\frac{g(t)e^{-at}}{b-a} \end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_4(t) = -\int^t \frac{g(s)e^{-as}}{b-a}\,ds$$

The particular solution is then

$$\begin{aligned}
y_p(t) &= C_4(t)e^{at} + C_5(t)e^{bt} \\
&= e^{at}\left[-\int^t \frac{g(s)e^{-as}}{b-a}\,ds\right] + e^{bt}\left[\int^t \frac{g(s)e^{-bs}}{b-a}\,ds\right] \\
&= -e^{at}\int^t \frac{g(s)e^{-as}}{b-a}\,ds + e^{bt}\int^t \frac{g(s)e^{-bs}}{b-a}\,ds \\
&= \int^t \left[-\frac{g(s)e^{at-as}}{b-a} + \frac{g(s)e^{bt-bs}}{b-a}\right] ds \\
&= \frac{1}{b-a}\int^t [e^{b(t-s)} - e^{a(t-s)}]g(s)\,ds \\
&= \frac{1}{b-a}\int_{t_0}^t [e^{b(t-s)} - e^{a(t-s)}]g(s)\,ds
\end{aligned}$$

The lower limit of integration is arbitrary and has been set to t_0 because that's when the initial conditions are given. Consequently, the general solution is

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= C_4e^{at} + C_5e^{bt} + \frac{1}{b-a}\int_{t_0}^t [e^{b(t-s)} - e^{a(t-s)}]g(s)\,ds.
\end{aligned}$$

Differentiate it with respect to t using the Leibnitz rule.

$$\begin{aligned}
y'(t) &= aC_4e^{at} + bC_5e^{bt} + \frac{1}{b-a}\int_{t_0}^t \frac{\partial}{\partial t}[e^{b(t-s)} - e^{a(t-s)}]g(s)\,ds + 1\cdot(e^0 - e^0)g(t) \\
&= aC_4e^{at} + bC_5e^{bt} + \frac{1}{b-a}\int_{t_0}^t [be^{b(t-s)} - ae^{a(t-s)}]g(s)\,ds
\end{aligned}$$

Apply the initial conditions now to determine C_4 and C_5.

$$\begin{aligned}
y(t_0) &= C_4e^{at_0} + C_5e^{bt_0} = 0 \\
y'(t_0) &= aC_4e^{at_0} + bC_5e^{bt_0} = 0
\end{aligned}$$

This system is only satisfied if $C_4 = 0$ and $C_5 = 0$. Therefore,

$$y(t) = \frac{1}{b-a}\int_{t_0}^t [e^{b(t-s)} - e^{a(t-s)}]g(s)\,ds.$$