Problem 25

Use the result of Problem 22 to find the solution of the initial value problem

$$L[y] = [D^2 - 2\lambda D + (\lambda^2 + \mu^2)]y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0.$$

Note that the roots of the characteristic equation are $\lambda \pm i\mu$.

Solution

Method Using Operator Factorization

The roots of the characteristic equation are $\lambda \pm i\mu$, so the operator in the ODE can be factored as follows.

$$[D - (\lambda + i\mu)][D - (\lambda - i\mu)]y = g(t)$$

Make the substitution, $u = [D - (\lambda - i\mu)]y$.

$$[D - (\lambda + i\mu)]u = g(t)$$

As a result, the second-order ODE has been reduced to a system of (decoupled) first-order ODEs.

$$\begin{cases} [D - (\lambda - i\mu)]y = u & \rightarrow \quad y' - (\lambda - i\mu)y = u(t) \\ [D - (\lambda + i\mu)]u = g & \rightarrow \quad u' - (\lambda + i\mu)u = g(t) \end{cases}$$

Solve the one for u first by using an integrating factor I_1.

$$I_1 = \exp\left[\int^t -(\lambda + i\mu)\, ds\right] = e^{-(\lambda+i\mu)t}$$

Multiply both sides of the ODE for u by I_1.

$$e^{-(\lambda+i\mu)t}u' - (\lambda + i\mu)e^{-(\lambda+i\mu)t}u = g(t)e^{-(\lambda+i\mu)t}$$

The left side can be written as $d/dt(I_1 u)$ by the product rule.

$$\frac{d}{dt}\left[e^{-(\lambda+i\mu)t}u\right] = g(t)e^{-(\lambda+i\mu)t}$$

Integrate both sides with respect to t.

$$e^{-(\lambda+i\mu)t}u = \int^t g(s)e^{-(\lambda+i\mu)s}\, ds + C_1$$

Multiply both sides by $e^{(\lambda+i\mu)t}$.

$$u(t) = e^{(\lambda+i\mu)t}\int^t g(s)e^{-(\lambda+i\mu)s}\, ds + C_1 e^{(\lambda+i\mu)t}$$

Substitute this result into the ODE for y.

$$y' - (\lambda - i\mu)y = e^{(\lambda+i\mu)t}\int^t g(s)e^{-(\lambda+i\mu)s}\, ds + C_1 e^{(\lambda+i\mu)t}$$

Use another integrating factor I_2 to solve it.

$$I_2 = \exp\left[\int^t -(\lambda - i\mu)\,ds\right] = e^{-(\lambda-i\mu)t}$$

Multiply both sides of the previous equation by I_2.

$$e^{-(\lambda-i\mu)t}y' - (\lambda - i\mu)e^{-(\lambda-i\mu)t}y = e^{-(\lambda-i\mu)t}e^{(\lambda+i\mu)t}\int^t g(s)e^{-(\lambda+i\mu)s}\,ds + C_1e^{-(\lambda-i\mu)t}e^{(\lambda+i\mu)t}$$

The left side can be written as $d/dt(I_2y)$ by the product rule.

$$\frac{d}{dt}\left[e^{-(\lambda-i\mu)t}y\right] = e^{2i\mu t}\int^t g(s)e^{-(\lambda+i\mu)s}\,ds + C_1e^{2i\mu t}$$

Integrate both sides with respect to t.

$$e^{-(\lambda-i\mu)t}y = \int^t e^{2i\mu q}\int^q g(s)e^{-(\lambda+i\mu)s}\,ds\,dq + \frac{C_1}{2i\mu}e^{2i\mu t} + C_2$$

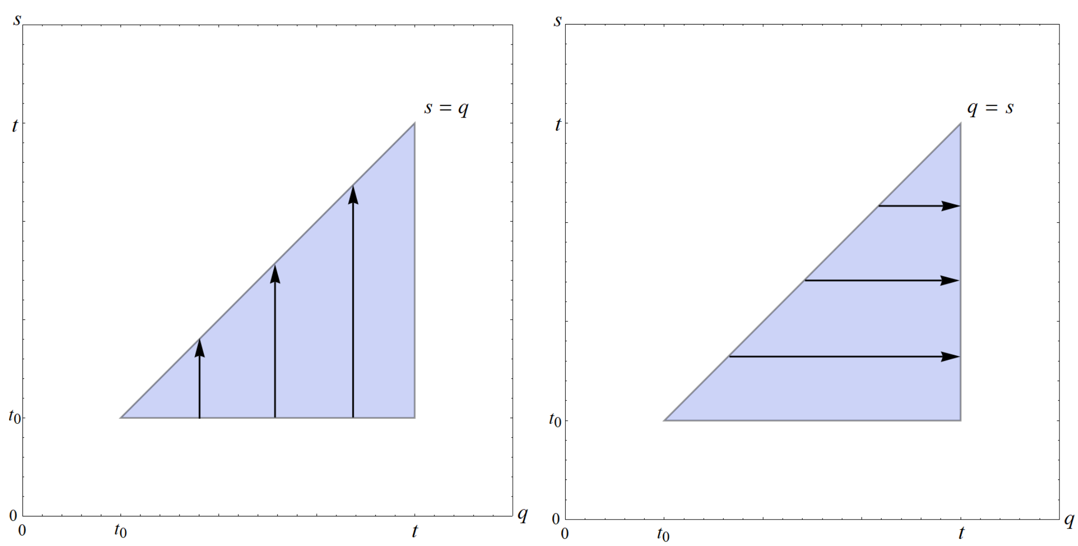
Multiply both sides by $e^{(\lambda-i\mu)t}$ and use a new constant C_3 for $C_1/(2i\mu)$.

$$\begin{aligned}
y(t) &= e^{(\lambda-i\mu)t}\int^t e^{2i\mu q}\int^q g(s)e^{-(\lambda+i\mu)s}\,ds\,dq + C_3e^{2i\mu t}e^{(\lambda-i\mu)t} + C_2e^{(\lambda-i\mu)t} \\
&= e^{\lambda t}e^{-i\mu t}\int^t\int^q e^{2i\mu q}g(s)e^{-\lambda s}e^{-i\mu s}\,ds\,dq + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t} \\
&= \int^t\int^q e^{\lambda(t-s)}e^{2i\mu q}e^{-i\mu t-i\mu s}g(s)\,ds\,dq + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t}
\end{aligned}$$

Since the initial conditions are given at $t = t_0$, the lower limits of integration will be set to t_0.

$$y(t) = \int_{t_0}^t\int_{t_0}^q e^{\lambda(t-s)}e^{2i\mu q}e^{-i\mu t-i\mu s}g(s)\,ds\,dq + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t}$$

The current mode of integration in the qs-plane is shown below on the left.



Integrate over the domain as shown on the right to switch the order of integration.

$$\begin{aligned}
y(t) &= \int_{t_0}^{t}\int_{s}^{t} e^{\lambda(t-s)}e^{2i\mu q}e^{-i\mu t-i\mu s}g(s)\,dq\,ds + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t} \\
&= \int_{t_0}^{t} e^{\lambda(t-s)}\left(\frac{1}{2i\mu}e^{2i\mu q}\bigg|_s^t\right)e^{-i\mu t-i\mu s}g(s)\,ds + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t} \\
&= \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\left(\frac{e^{2i\mu t}-e^{2i\mu s}}{2i}\right)e^{-i\mu t-i\mu s}g(s)\,ds + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t} \\
&= \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\left(\frac{e^{i\mu t-i\mu s}-e^{i\mu s-i\mu t}}{2i}\right)g(s)\,ds + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t} \\
&= \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\sin(\mu t-\mu s)g(s)\,ds + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t}
\end{aligned}$$

As a result, the general solution is

$$y(t) = \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\sin[\mu(t-s)]g(s)\,ds + C_3e^{(\lambda+i\mu)t} + C_2e^{(\lambda-i\mu)t}.$$

Use the Leibnitz rule,

$$\frac{d}{dt}\int_{j(t)}^{h(t)} f(t,s)\,ds = \int_{j(t)}^{h(t)}\frac{\partial}{\partial t}f(t,s)\,ds + \frac{dh}{dt}f[t,h(t)] - \frac{dj}{dt}f[t,j(t)],$$

to differentiate the general solution.

$$\begin{aligned}
y'(t) &= \frac{1}{\mu}\int_{t_0}^{t}\frac{\partial}{\partial t}e^{\lambda(t-s)}\sin[\mu(t-s)]g(s)\,ds + 1\cdot e^0(\sin 0)g(t) + C_3(\lambda+i\mu)e^{(\lambda+i\mu)t} + C_2(\lambda-i\mu)e^{(\lambda-i\mu)t} \\
&= \frac{1}{\mu}\int_{t_0}^{t}\left\{\lambda e^{\lambda(t-s)}\sin[\mu(t-s)] + \mu e^{\lambda(t-s)}\cos[\mu(t-s)]\right\}g(s)\,ds + C_3(\lambda+i\mu)e^{(\lambda+i\mu)t} + C_2(\lambda-i\mu)e^{(\lambda-i\mu)t}
\end{aligned}$$

Apply the initial conditions now to determine C_2 and C_3.

$$\begin{aligned}
y(t_0) &= C_3e^{(\lambda+i\mu)t_0} + C_2e^{(\lambda-i\mu)t_0} = 0 \\
y'(t_0) &= C_3(\lambda+i\mu)e^{(\lambda+i\mu)t_0} + C_2(\lambda-i\mu)e^{(\lambda-i\mu)t_0} = 0
\end{aligned}$$

This system is only satisfied if $C_2 = 0$ and $C_3 = 0$. Therefore,

$$y(t) = \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\sin[\mu(t-s)]g(s)\,ds.$$

Method Using Variation of Parameters

$$L[y] = [D^2 - 2\lambda D + (\lambda^2 + \mu^2)]y = g(t), \qquad y(t_0) = 0, \quad y'(t_0) = 0$$

Distribute the operator to obtain a second-order ODE.

$$y'' - 2\lambda y' + (\lambda^2 + \mu^2)y = g(t)$$

Because the ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' - 2\lambda y_c' + (\lambda^2 + \mu^2)y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} - 2\lambda(re^{rt}) + (\lambda^2 + \mu^2)(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 - 2\lambda r + (\lambda^2 + \mu^2) = 0$$

$$r = \frac{2\lambda \pm \sqrt{4\lambda^2 - 4(\lambda^2 + \mu^2)}}{2} = \frac{2\lambda \pm \sqrt{-4\mu^2}}{2} = \frac{2\lambda \pm 2i\mu}{2} = \lambda \pm i\mu$$

$$r = \{\lambda - i\mu, \lambda + i\mu\}$$

Two solutions to equation (1) are then $y_c = e^{(\lambda - i\mu)t}$ and $y_c = e^{(\lambda + i\mu)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_4 e^{(\lambda - i\mu)t} + C_5 e^{(\lambda + i\mu)t} \\
&= C_4 e^{\lambda t - i\mu t} + C_5 e^{\lambda t + i\mu t} \\
&= C_4 e^{\lambda t} e^{-i\mu t} + C_5 e^{\lambda t} e^{i\mu t} \\
&= C_4 e^{\lambda t}[\cos(-\mu t) + i\sin(-\mu t)] + C_5 e^{\lambda t}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_4 e^{\lambda t}[\cos(\mu t) - i\sin(\mu t)] + C_5 e^{\lambda t}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_4 e^{\lambda t}\cos\mu t - iC_4 e^{\lambda t}\sin\mu t + C_5 e^{\lambda t}\cos\mu t + iC_5 e^{\lambda t}\sin\mu t \\
&= (C_4 + C_5)e^{\lambda t}\cos\mu t + (-iC_4 + iC_5)e^{\lambda t}\sin\mu t \\
&= C_6 e^{\lambda t}\cos\mu t + C_7 e^{\lambda t}\sin\mu t
\end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_6(t)e^{\lambda t}\cos\mu t + C_7(t)e^{\lambda t}\sin\mu t$$

It satisfies the following ODE.

$$y_p'' - 2\lambda y_p' + (\lambda^2 + \mu^2) y_p = g(t)$$

Substitute the previous formula for $y_p(t)$.

$$[C_6(t)e^{\lambda t}\cos\mu t + C_7(t)e^{\lambda t}\sin\mu t]'' - 2\lambda[C_6(t)e^{\lambda t}\cos\mu t + C_7(t)e^{\lambda t}\sin\mu t]' \\ + (\lambda^2+\mu^2)[C_6(t)e^{\lambda t}\cos\mu t + C_7(t)e^{\lambda t}\sin\mu t] = g(t)$$

Evaluate the derivatives.

$$[C_6'(t)e^{\lambda t}\cos\mu t + \lambda C_6(t)e^{\lambda t}\cos\mu t - \mu C_6(t)e^{\lambda t}\sin\mu t \\ + C_7'(t)e^{\lambda t}\sin\mu t + \lambda C_7(t)e^{\lambda t}\sin\mu t + \mu C_7(t)e^{\lambda t}\cos\mu t]' \\ - 2\lambda[C_6'(t)e^{\lambda t}\cos\mu t + \lambda C_6(t)e^{\lambda t}\cos\mu t - \mu C_6(t)e^{\lambda t}\sin\mu t \\ + C_7'(t)e^{\lambda t}\sin\mu t + \lambda C_7(t)e^{\lambda t}\sin\mu t + \mu C_7(t)e^{\lambda t}\cos\mu t] \\ + (\lambda^2+\mu^2)[C_6(t)e^{\lambda t}\cos\mu t + C_7(t)e^{\lambda t}\sin\mu t] = g(t)$$

$$[C_6''(t)e^{\lambda t}\cos\mu t + \lambda C_6'(t)e^{\lambda t}\cos\mu t - \mu C_6'(t)e^{\lambda t}\sin\mu t + \lambda C_6'(t)e^{\lambda t}\cos\mu t + \lambda^2 C_6(t)e^{\lambda t}\cos\mu t \\ - \mu\lambda C_6(t)e^{\lambda t}\sin\mu t - \mu C_6'(t)e^{\lambda t}\sin\mu t - \mu\lambda C_6(t)e^{\lambda t}\sin\mu t - \mu^2 C_6(t)e^{\lambda t}\cos\mu t \\ + C_7''(t)e^{\lambda t}\sin\mu t + \lambda C_7'(t)e^{\lambda t}\sin\mu t + \mu C_7'(t)e^{\lambda t}\cos\mu t + \lambda C_7'(t)e^{\lambda t}\sin\mu t \\ + \lambda^2 C_7(t)e^{\lambda t}\sin\mu t + \mu\lambda C_7(t)e^{\lambda t}\cos\mu t + \mu C_7'(t)e^{\lambda t}\cos\mu t + \mu\lambda C_7(t)e^{\lambda t}\cos\mu t - \mu^2 C_7(t)e^{\lambda t}\sin\mu t] \\ - 2\lambda[C_6'(t)e^{\lambda t}\cos\mu t + \lambda C_6(t)e^{\lambda t}\cos\mu t - \mu C_6(t)e^{\lambda t}\sin\mu t \\ + C_7'(t)e^{\lambda t}\sin\mu t + \lambda C_7(t)e^{\lambda t}\sin\mu t + \mu C_7(t)e^{\lambda t}\cos\mu t] \\ + (\lambda^2+\mu^2)[C_6(t)e^{\lambda t}\cos\mu t + C_7(t)e^{\lambda t}\sin\mu t] = g(t)$$

Simplify the left side.

$$e^{\lambda t}(\cos\mu t)C_6''(t) - 2\mu e^{\lambda t}(\sin\mu t)C_6'(t) + e^{\lambda t}(\sin\mu t)C_7''(t) + 2\mu e^{\lambda t}(\cos\mu t)C_7'(t) = g(t)$$

Divide both sides by $e^{\lambda t}$.

$$(\cos\mu t)C_6''(t) - 2\mu(\sin\mu t)C_6'(t) + (\sin\mu t)C_7''(t) + 2\mu(\cos\mu t)C_7'(t) = g(t)e^{-\lambda t}$$

If we set

$$(\cos\mu t)C_6''(t) - \mu(\sin\mu t)C_6'(t) + (\sin\mu t)C_7''(t) + \mu(\cos\mu t)C_7'(t) = 0, \tag{2}$$

then the previous equation reduces to

$$-\mu(\sin\mu t)C_6'(t) + \mu(\cos\mu t)C_7'(t) = g(t)e^{-\lambda t}. \tag{3}$$

The aim now is to solve this system of two equations for $C_6(t)$ and $C_7(t)$. Start by rewriting equation (2).

$$\frac{d}{dt}[(\cos\mu t)C_6'(t)] + \frac{d}{dt}[(\sin\mu t)C_7'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$(\cos\mu t)C_6'(t) + (\sin\mu t)C_7'(t) = 0$$

Solve for $C_6'(t)$.

$$C_6'(t) = -\frac{\sin \mu t}{\cos \mu t} C_7'(t) \tag{4}$$

Plug this formula into equation (3).

$$\begin{aligned}
-\mu(\sin \mu t)\left[-\frac{\sin \mu t}{\cos \mu t} C_7'(t)\right] + \mu(\cos \mu t) C_7'(t) &= g(t)e^{-\lambda t} \\
\mu C_7'(t)\left(\frac{\sin^2 \mu t}{\cos \mu t} + \cos \mu t\right) &= g(t)e^{-\lambda t} \\
\mu C_7'(t)\left(\frac{\sin^2 \mu t + \cos^2 \mu t}{\cos \mu t}\right) &= g(t)e^{-\lambda t} \\
\mu C_7'(t)\left(\frac{1}{\cos \mu t}\right) &= g(t)e^{-\lambda t}
\end{aligned}$$

Multiply both sides by $(\cos \mu t)/\mu$.

$$C_7'(t) = \frac{1}{\mu} g(t) e^{-\lambda t} \cos \mu t$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_7(t) = \int^t \frac{1}{\mu} g(s) e^{-\lambda s} \cos \mu s \, ds$$

Substitute the previous formula for $C_7'(t)$ into equation (4).

$$\begin{aligned}
C_6'(t) &= -\frac{\sin \mu t}{\cos \mu t}\left[\frac{1}{\mu} g(t) e^{-\lambda t} \cos \mu t\right] \\
&= -\frac{1}{\mu} g(t) e^{-\lambda t} \sin \mu t
\end{aligned}$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$C_6(t) = -\int^t \frac{1}{\mu} g(s) e^{-\lambda s} \sin \mu s \, ds$$

The particular solution is then

$$\begin{aligned}
y_p(t) &= C_6(t) e^{\lambda t} \cos \mu t + C_7(t) e^{\lambda t} \sin \mu t \\
&= e^{\lambda t} \cos \mu t \left[-\int^t \frac{1}{\mu} g(s) e^{-\lambda s} \sin \mu s \, ds\right] + e^{\lambda t} \sin \mu t \left[\int^t \frac{1}{\mu} g(s) e^{-\lambda s} \cos \mu s \, ds\right] \\
&= -e^{\lambda t} \cos \mu t \int^t \frac{1}{\mu} g(s) e^{-\lambda s} \sin \mu s \, ds + e^{\lambda t} \sin \mu t \int^t \frac{1}{\mu} g(s) e^{-\lambda s} \cos \mu s \, ds \\
&= \int^t \left[-\frac{1}{\mu} g(s) e^{\lambda t - \lambda s} \cos \mu t \sin \mu s + \frac{1}{\mu} g(s) e^{\lambda t - \lambda s} \sin \mu t \cos \mu s\right] ds \\
&= \frac{1}{\mu} \int^t e^{\lambda(t-s)} (\sin \mu t \cos \mu s - \cos \mu t \sin \mu s) g(s) \, ds \\
&= \frac{1}{\mu} \int^t e^{\lambda(t-s)} \sin(\mu t - \mu s) g(s) \, ds \\
&= \frac{1}{\mu} \int_{t_0}^t e^{\lambda(t-s)} \sin[\mu(t-s)] g(s) \, ds.
\end{aligned}$$

The lower limit of integration is arbitrary and has been set to t_0 because that's when the initial conditions are given. Consequently, the general solution is

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_6 e^{\lambda t}\cos\mu t + C_7 e^{\lambda t}\sin\mu t + \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\sin[\mu(t-s)]g(s)\,ds. \end{aligned}$$

Differentiate it with respect to t using the Leibnitz rule.

$$y'(t) = C_6(\lambda e^{\lambda t}\cos\mu t - \mu e^{\lambda t}\sin\mu t) + C_7(\lambda e^{\lambda t}\sin\mu t + \mu e^{\lambda t}\cos\mu t) + \frac{1}{\mu}\int_{t_0}^{t}\frac{\partial}{\partial t}e^{\lambda(t-s)}\sin[\mu(t-s)]g(s)\,ds$$

Apply the initial conditions now to determine C_6 and C_7.

$$\begin{aligned} y(t_0) &= C_6 e^{\lambda t_0}\cos\mu t_0 + C_7 e^{\lambda t_0}\sin\mu t_0 = 0 \\ y'(t_0) &= C_6(\lambda e^{\lambda t_0}\cos\mu t_0 - \mu e^{\lambda t_0}\sin\mu t_0) + C_7(\lambda e^{\lambda t_0}\sin\mu t_0 + \mu e^{\lambda t_0}\cos\mu t_0) = 0 \end{aligned}$$

This system is only satisfied if $C_6 = 0$ and $C_7 = 0$. Therefore,

$$y(t) = \frac{1}{\mu}\int_{t_0}^{t} e^{\lambda(t-s)}\sin[\mu(t-s)]g(s)\,ds.$$