Problem 30

In each of Problems 29 through 32, use the method outlined in Problem 28 to solve the given differential equation.

$$t^2y'' + 7ty' + 5y = t, \quad t > 0; \qquad y_1(t) = t^{-1}$$

Solution

Since one solution is known, the method of reduction of order can be applied to determine the general solution. Substitute $y(t) = c(t)t^{-1}$ into the ODE and solve the resulting ODE for $c(t)$.

$$t^2[c(t)t^{-1}]'' + 7t[c(t)t^{-1}]' + 5[c(t)t^{-1}] = t$$

Evaluate the derivatives.

$$t^2[c'(t)t^{-1} - c(t)t^{-2}]' + 7t[c'(t)t^{-1} - c(t)t^{-2}] + 5[c(t)t^{-1}] = t$$

$$t^2[c''(t)t^{-1} - c'(t)t^{-2} - c'(t)t^{-2} + 2c(t)t^{-3}] + 7t[c'(t)t^{-1} - c(t)t^{-2}] + 5[c(t)t^{-1}] = t$$

$$c''(t)t - c'(t) - c'(t) + \cancel{2c(t)t^{-1}} + 7c'(t) - \cancel{7c(t)t^{-1}} + \cancel{5c(t)t^{-1}} = t$$

$$tc''(t) + 5c'(t) = t$$

Divide both sides by t.

$$c''(t) + \frac{5}{t}c'(t) = 1$$

Use an integrating factor I to solve the ODE.

$$I = \exp\left(\int^t \frac{5}{s}\,ds\right) = e^{5\ln t} = e^{\ln t^5} = t^5$$

Multiply both sides of the previous equation by I.

$$t^5c''(t) + 5t^4c'(t) = t^5$$

The left side can be written as $d/dt[Ic'(t)]$ by the product rule.

$$\frac{d}{dt}[t^5c'(t)] = t^5$$

Integrate both sides with respect to t.

$$t^5c'(t) = \frac{t^6}{6} + C_1$$

Divide both sides by t^5.

$$c'(t) = \frac{t}{6} + C_1t^{-5}$$

Integrate both sides with respect to t once more.

$$c(t) = \frac{t^2}{12} - \frac{C_1}{4}t^{-4} + C_2$$

Therefore, using a new constant C_3 for $-C_1/4$,

$$\begin{aligned} y(t) &= c(t)t^{-1} \\ &= \left(\frac{t^2}{12} + C_3 t^{-4} + C_2\right) t^{-1} \\ &= \frac{t}{12} + C_3 t^{-5} + C_2 t^{-1}. \end{aligned}$$

The terms containing C_3 and C_2 are the second and first solutions, respectively, to the associated homogeneous equation, and the first term is the particular solution.