Problem 31

In each of Problems 29 through 32, use the method outlined in Problem 28 to solve the given differential equation.

$$ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0; \qquad y_1(t) = 1+t \qquad \text{(see Problem 15)}$$

Solution

Since one solution is known, the method of reduction of order can be applied to determine the general solution. Substitute $y(t) = c(t)(1+t)$ into the ODE and solve the resulting ODE for $c(t)$.

$$t[c(t)(1+t)]'' - (1+t)[c(t)(1+t)]' + [c(t)(1+t)] = t^2e^{2t}$$

Evaluate the derivatives.

$$t[c'(t)(1+t) + c(t)]' - (1+t)[c'(t)(1+t) + c(t)] + [c(t)(1+t)] = t^2e^{2t}$$

$$t[c''(t)(1+t) + c'(t) + c'(t)] - (1+t)[c'(t)(1+t) + c(t)] + [c(t)(1+t)] = t^2e^{2t}$$

$$t(1+t)c''(t) + 2tc'(t) - (1+t)^2c'(t) - \cancel{(1+t)c(t)} + \cancel{c(t)(1+t)} = t^2e^{2t}$$

$$t(1+t)c''(t) - (1+t^2)c'(t) = t^2e^{2t}$$

Divide both sides by $t(1+t)$.

$$c''(t) - \frac{1+t^2}{t(1+t)}c'(t) = \frac{t}{1+t}e^{2t}$$

Use an integrating factor I to solve this ODE.

$$I = \exp\left[\int^t -\frac{1+s^2}{s(1+s)}\,ds\right] = \exp\left[\int^t \left(-1 - \frac{1}{s} + \frac{2}{1+s}\right)ds\right] = e^{-t-\ln t+2\ln(1+t)} = e^{-t}t^{-1}(1+t)^2$$

Multiply both sides of the previous equation by I.

$$\frac{(1+t)^2}{t}e^{-t}c''(t) - \frac{(1+t^2)(1+t)}{t^2}e^{-t}c'(t) = (1+t)e^t$$

The left side can be written as $d/dt[Ic'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\frac{(1+t)^2}{t}e^{-t}c'(t)\right] = (1+t)e^t$$

Integrate both sides with respect to t.

$$\begin{aligned}
\frac{(1+t)^2}{t}e^{-t}c'(t) &= \int^t (1+s)e^s\,ds + C_1 \\
&= \int^t e^s\,ds + \int^t se^s\,ds + C_1 \\
&= e^t + \int^t s\frac{d}{ds}(e^s)\,ds + C_1 \\
&= e^t + \left[s(e^s)\Big|^t - \int^t (1)e^s\,ds\right] + C_1 \\
&= e^t + (te^t - e^t) + C_1 \\
&= te^t + C_1
\end{aligned}$$

Divide both sides by I.

$$c'(t) = \frac{t^2}{(1+t)^2}e^{2t} + C_1\frac{te^t}{(1+t)^2}$$

Integrate both sides with respect to t once more.

$$c(t) = \frac{t-1}{2(1+t)}e^{2t} + C_1\frac{e^t}{1+t} + C_2$$

Therefore,

$$\begin{aligned} y(t) &= c(t)(1+t) \\ &= \left[\frac{t-1}{2(1+t)}e^{2t} + C_1\frac{e^t}{1+t} + C_2\right](1+t) \\ &= \frac{t-1}{2}e^{2t} + C_1e^t + C_2(1+t). \end{aligned}$$

The terms containing C_1 and C_2 are the second and first solutions, respectively, to the associated homogeneous equation, and the first term is the particular solution.