Problem 32

In each of Problems 29 through 32, use the method outlined in Problem 28 to solve the given differential equation.

$$(1-t)y'' + ty' - y = 2(t-1)^2 e^{-t}, \quad 0 < t < 1; \qquad y_1(t) = e^t \qquad \text{(see Problem 16)}$$

Solution

Since one solution is known, the method of reduction of order can be applied to determine the general solution. Substitute $y(t) = c(t)e^t$ into the ODE and solve the resulting ODE for $c(t)$.

$$(1-t)[c(t)e^t]'' + t[c(t)e^t]' - [c(t)e^t] = 2(t-1)^2 e^{-t}$$

Evaluate the derivatives.

$$(1-t)[c'(t)e^t + c(t)e^t]' + t[c'(t)e^t + c(t)e^t] - [c(t)e^t] = 2(t-1)^2 e^{-t}$$

$$(1-t)[c''(t)e^t + c'(t)e^t + c'(t)e^t + c(t)e^t] + t[c'(t)e^t + c(t)e^t] - [c(t)e^t] = 2(t-1)^2 e^{-t}$$

Simplify the left side.

$$(1-t)e^t c''(t) + (2-t)e^t c'(t) = 2(1-t)^2 e^{-t}$$

Divide both sides by $(1-t)e^t$.

$$c''(t) + \frac{2-t}{1-t}c'(t) = 2(1-t)e^{-2t}$$

Use an integrating factor I to solve this ODE.

$$I = \exp\left(\int^t \frac{2-s}{1-s}\,ds\right) = \exp\left(2\int^t \frac{ds}{1-s} - \int^t \frac{s}{1-s}\,ds\right) = e^{-2\ln(1-t)+t+\ln(1-t)} = (1-t)^{-1}e^t$$

Multiply both sides of the previous equation by I.

$$\frac{e^t}{1-t}c''(t) + \frac{2-t}{(1-t)^2}e^t c'(t) = 2e^{-t}$$

The left side can be written as $d/dt[Ic'(t)]$ by the product rule.

$$\frac{d}{dt}\left[\frac{e^t}{1-t}c'(t)\right] = 2e^{-t}$$

Integrate both sides with respect to t.

$$\frac{e^t}{1-t}c'(t) = -2e^{-t} + C_1$$

Divide both sides by I.

$$c'(t) = -2(1-t)e^{-2t} + C_1(1-t)e^{-t}$$

Integrate both sides with respect to t once more.

$$c(t) = \frac{1}{2}(1-2t)e^{-2t} + C_1 te^{-t} + C_2$$

Therefore,

$$\begin{aligned} y(t) &= c(t)e^t \\ &= \left[\frac{1}{2}(1-2t)e^{-2t} + C_1 te^{-t} + C_2\right] e^t \\ &= \frac{1}{2}(1-2t)e^{-t} + C_1 t + C_2 e^t. \end{aligned}$$

The terms containing C_1 and C_2 are the second and first solutions, respectively, to the associated homogeneous equation, and the first term is the particular solution.