Problem 8

In each of Problems 5 through 12, find the general solution of the given differential equation. In Problems 11 and 12, g is an arbitrary continuous function.

$$y'' + 4y = 3\csc 2t, \qquad 0 < t < \pi/2$$

Solution

Because this ODE is linear, the general solution can be expressed as a sum of the complementary solution $y_c(t)$ and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$y_c'' + 4y_c = 0 \tag{1}$$

This is a homogeneous ODE with constant coefficients, so the solution is of the form $y_c = e^{rt}$.

$$y_c = e^{rt} \quad \rightarrow \quad y_c' = re^{rt} \quad \rightarrow \quad y_c'' = r^2e^{rt}$$

Substitute these expressions into the ODE.

$$r^2e^{rt} + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 4 = 0$$

$$r = \{-2i, 2i\}$$

Two solutions to equation (1) are then $y_c = e^{-2it}$ and $y_c = e^{2it}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
y_c(t) &= C_1e^{-2it} + C_2e^{2it} \\
&= C_1[\cos(-2t) + i\sin(-2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1[\cos(2t) - i\sin(2t)] + C_2[\cos(2t) + i\sin(2t)] \\
&= C_1\cos 2t - iC_1\sin 2t + C_2\cos 2t + iC_2\sin 2t \\
&= (C_1 + C_2)\cos 2t + (-iC_1 + iC_2)\sin 2t \\
&= C_3\cos 2t + C_4\sin 2t
\end{aligned}$$

According to the method of variation of parameters, the particular solution is obtained by allowing the parameters in $y_c(t)$ to vary.

$$y_p(t) = C_3(t)\cos 2t + C_4(t)\sin 2t$$

It satisfies the following ODE.

$$y_p'' + 4y_p = 3\csc 2t$$

Substitute the previous formula in for $y_p(t)$.

$$[C_3(t)\cos 2t + C_4(t)\sin 2t]'' + 4[C_3(t)\cos 2t + C_4(t)\sin 2t] = 3\csc 2t$$

Evaluate the derivatives.

$$[C_3'(t)\cos 2t - 2C_3(t)\sin 2t + C_4'(t)\sin 2t + 2C_4(t)\cos 2t]' + 4[C_3(t)\cos 2t + C_4(t)\sin 2t] = 3\csc 2t$$

$$[C_3''(t)\cos 2t - 2C_3'(t)\sin 2t - 2C_3'(t)\sin 2t - \cancel{4C_3(t)\cos 2t} + C_4''(t)\sin 2t + 2C_4'(t)\cos 2t + 2C_4'(t)\cos 2t - \cancel{4C_4(t)\sin 2t}] + \cancel{4C_3(t)\cos 2t} + \cancel{4C_4(t)\sin 2t} = 3\csc 2t$$

$$C_3''(t)\cos 2t - 4C_3'(t)\sin 2t + C_4''(t)\sin 2t + 4C_4'(t)\cos 2t = 3\csc 2t$$

If we set

$$C_3''(t)\cos 2t - 4C_3'(t)\sin 2t = 0, \tag{2}$$

then the previous equation reduces to

$$C_4''(t)\sin 2t + 4C_4'(t)\cos 2t = 3\csc 2t. \tag{3}$$

The aim now is to solve this system of two equations for $C_3(t)$ and $C_4(t)$. Start by dividing equation (2) by $\cos 2t$.

$$C_3''(t) - 4\frac{\sin 2t}{\cos 2t}C_3'(t) = 0$$

Use an integrating factor I_1 to solve it.

$$I_1 = \exp\left(\int^t -4\frac{\sin 2s}{\cos 2s}\,ds\right) = e^{2\ln\cos 2t} = e^{\ln\cos^2 2t} = \cos^2 2t$$

Multiply both sides of the previous equation by I_1.

$$(\cos^2 2t)C_3''(t) - (4\sin 2t\cos 2t)C_3'(t) = 0$$

The left side can be written as $d/dt[I_1 C_3'(t)]$ by the product rule.

$$\frac{d}{dt}[(\cos^2 2t)C_3'(t)] = 0$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$(\cos^2 2t)C_3'(t) = 0$$

Divide both sides by $\cos^2 2t$.

$$C_3'(t) = 0$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_3(t) = 0$$

Divide both sides of equation (3) by $\sin 2t$.

$$C_4''(t) + 4\frac{\cos 2t}{\sin 2t}C_4'(t) = 3\csc^2 2t$$

Use an integrating factor I_2 to solve it.

$$I_2 = \exp\left(\int^t 4\frac{\cos 2s}{\sin 2s}\,ds\right) = e^{2\ln\sin 2t} = e^{\ln\sin^2 2t} = \sin^2 2t$$

Multiply both sides of the previous equation by I_2.

$$(\sin^2 2t)C_4''(t) + (4\cos 2t \sin 2t)C_4'(t) = 3.$$

The left side can be written as $d/dt[I_2C_4'(t)]$ by the product rule.

$$\frac{d}{dt}[(\sin^2 2t)C_4'(t)] = 3$$

Integrate both sides with respect to t, setting the integration constant to zero.

$$(\sin^2 2t)C_4'(t) = 3t$$

Divide both sides by $\sin^2 2t$.

$$C_4'(t) = 3t\csc^2 2t$$

Integrate both sides with respect to t once more, setting the integration constant to zero.

$$C_4(t) = \int^t 3s\csc^2 2s\,ds$$

Make the following substitution.

$$u = 2s \quad \rightarrow \quad \frac{u}{2} = s$$

$$du = 2\,ds \quad \rightarrow \quad \frac{du}{2} = ds$$

As a result,

$$\begin{aligned}
C_4(t) &= \int^{2t} 3\frac{u}{2}\csc^2 u\,\frac{du}{2} \\
&= \frac{3}{4}\int^{2t} u\csc^2 u\,du \\
&= \frac{3}{4}\int^{2t} u\frac{d}{du}(-\cot u)\,du \\
&= \frac{3}{4}\left[u(-\cot u)\Big|^{2t} - \int^{2t}(1)(-\cot u)\,du\right] \\
&= \frac{3}{4}\left(-2t\cot 2t + \int^{2t}\cot u\,du\right) \\
&= \frac{3}{4}(-2t\cot 2t + \ln|\sin 2t|).
\end{aligned}$$

Since $0 < t < \pi/2$, the absolute value sign can be removed. The particular solution is then

$$\begin{aligned}
y_p(t) &= C_3(t)\cos 2t + C_4(t)\sin 2t \\
&= \frac{3}{4}[-2t\cos 2t + (\sin 2t)\ln\sin 2t].
\end{aligned}$$

Therefore,

$$\begin{aligned}
y(t) &= y_c(t) + y_p(t) \\
&= C_3\cos 2t + C_4\sin 2t + \frac{3}{4}[-2t\cos 2t + (\sin 2t)\ln\sin 2t].
\end{aligned}$$