Problem 13

A certain vibrating system satisfies the equation $u'' + \gamma u' + u = 0$. Find the value of the damping coefficient γ for which the quasi period of the damped motion is 50% greater than the period of the corresponding undamped motion.

Solution

Begin by solving the ODE for u.

$$u'' + \gamma u' + u = 0$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2 e^{rt} + \gamma(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + \gamma r + 1 = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2}$$

Because we expect there to be a quasi period, $\gamma^2 - 4 < 0$. Write it as $4 - \gamma^2$ under the square root.

$$r = \frac{-\gamma \pm i\sqrt{4 - \gamma^2}}{2}$$

$$r = \left\{-\frac{\gamma}{2} - i\mu, -\frac{\gamma}{2} + i\mu\right\}$$

Two solutions to the ODE are $u = e^{(-\gamma/2 - i\mu)t}$ and $u = e^{(-\gamma/2 + i\mu)t}$. By the principle of superposition, the general solution for u is a linear combination of the two.

$$\begin{aligned}
u(t) &= C_1 e^{(-\gamma/2 - i\mu)t} + C_2 e^{(-\gamma/2 + i\mu)t} \\
&= C_1 e^{-\gamma t/2 - i\mu t} + C_2 e^{-\gamma t/2 + i\mu t} \\
&= C_1 e^{-\gamma t/2} e^{-i\mu t} + C_2 e^{-\gamma t/2} e^{i\mu t} \\
&= C_1 e^{-\gamma t/2}[\cos(-\mu t) + i\sin(-\mu t)] + C_2 e^{-\gamma t/2}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-\gamma t/2}[\cos(\mu t) - i\sin(\mu t)] + C_2 e^{-\gamma t/2}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-\gamma t/2}\cos\mu t - iC_1 e^{-\gamma t/2}\sin\mu t + C_2 e^{-\gamma t/2}\cos\mu t + iC_2 e^{-\gamma t/2}\sin\mu t \\
&= (C_1 + C_2)e^{-\gamma t/2}\cos\mu t + (-iC_1 + iC_2)e^{-\gamma t/2}\sin\mu t \\
&= C_3 e^{-\gamma t/2}\cos\mu t + C_4 e^{-\gamma t/2}\sin\mu t \\
&= e^{-\gamma t/2}(C_3\cos\mu t + C_4\sin\mu t)
\end{aligned}$$

Since C_3 and C_4 are arbitrary, we can introduce an amplitude A and a phase δ to write the two sinusoidal terms as one.

$$\begin{aligned} u(t) &= e^{-\gamma t/2}(A\cos\delta\cos\mu t + A\sin\delta\sin\mu t) \\ &= Ae^{-\gamma t/2}\cos(\mu t - \delta) \\ &= Ae^{-\gamma t/2}\cos\left(\frac{\sqrt{4-\gamma^2}}{2}t - \delta\right) \end{aligned}$$

The quasi period of $u(t)$ is

$$T_d = \frac{2\pi}{\mu} = \frac{2\pi}{\frac{\sqrt{4-\gamma^2}}{2}}.$$

Comparing the ODE with the general equation of motion for a mass connected to a spring and viscous damper,

$$u'' + \gamma u' + u = 0 \qquad mu'' + cu' + ku = 0$$

we see that k and m are both 1. The angular frequency of the corresponding undamped motion is then $\omega = \sqrt{k/m} = 1$, and the period is $T = 2\pi/\omega = 2\pi$. If the period of the damped motion is 50% greater than this, then

$$\begin{aligned} T_d &= \frac{3}{2}T \\ \frac{2\pi}{\frac{\sqrt{4-\gamma^2}}{2}} &= \frac{3}{2}(2\pi). \end{aligned}$$

Therefore, solving for γ,

$$\gamma = \frac{2\sqrt{5}}{3} \approx 1.49.$$