Problem 15

Show that the solution of the initial value problem

$$mu'' + \gamma u' + ku = 0, \qquad u(t_0) = u_0, \quad u'(t_0) = u_0'$$

can be expressed as the sum $u = v + w$, where v satisfies the initial conditions $v(t_0) = u_0$, $v'(t_0) = 0$, w satisfies the initial conditions $w(t_0) = 0$, $w'(t_0) = u_0'$, and both v and w satisfy the same differential equation as u. This is another instance of superposing solutions of simpler problems to obtain the solution of a more general problem.

Solution

Make the substitution, $u = v + w$, in the ODE.

$$m(v+w)'' + \gamma(v+w)' + k(v+w) = 0$$

Evaluate the derivatives.

$$m(v'' + w'') + \gamma(v' + w') + k(v+w) = 0$$

Rearrange the terms.

$$mv'' + \gamma v' + kv + mw'' + \gamma w' + kw = 0$$

If we set

$$mv'' + \gamma v' + kv = 0, \tag{1}$$

then the previous equation reduces to

$$mw'' + \gamma w' + kw = 0. \tag{2}$$

As a result, v and w satisfy the same ODE as u. Now apply the substitution to the initial conditions.

$$\begin{aligned} u(t_0) = u_0 \qquad &\rightarrow \qquad v(t_0) + w(t_0) = u_0 \\ u'(t_0) = u_0' \qquad &\rightarrow \qquad v'(t_0) + w'(t_0) = u_0' \end{aligned}$$

If we set $v(t_0) = u_0$ and $w'(t_0) = u_0'$, then the other initial conditions must satisfy

$$\begin{aligned} v(t_0) + w(t_0) = u_0 \quad &\rightarrow \quad u_0 + w(t_0) = u_0 \quad \rightarrow \quad w(t_0) = 0 \\ v'(t_0) + w'(t_0) = u_0' \quad &\rightarrow \quad v'(t_0) + u_0' = u_0' \quad \rightarrow \quad v'(t_0) = 0. \end{aligned}$$

Therefore, the general solution to

$$mu'' + \gamma u' + ku = 0, \qquad u(t_0) = u_0, \quad u'(t_0) = u_0'$$

can be found by splitting up the problem into

$$\begin{aligned} mv'' + \gamma v' + kv = 0, \qquad &v(t_0) = u_0, \quad v'(t_0) = 0 \\ mw'' + \gamma w' + kw = 0, \qquad &w(t_0) = 0, \quad w'(t_0) = u_0' \end{aligned}$$

and then adding v and w together. This works because the initial value problem for u is linear.