Problem 20

Assume that the system described by the equation $mu'' + \gamma u' + ku = 0$ is critically damped and that the initial conditions are $u(0) = u_0$, $u'(0) = v_0$. If $v_0 = 0$, show that $u \to 0$ as $t \to \infty$ but that u is never zero. If u_0 is positive, determine a condition on v_0 that will ensure that the mass passes through its equilibrium position after it is released.

Solution

Suppose that the motion is critically damped. Then the ratio of γ^2 to $4km$ is 1, and $\gamma = \sqrt{4km}$. As a result, the ODE becomes

$$mu'' + \sqrt{4km}u' + ku = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \to \quad u' = re^{rt} \quad \to \quad u'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2 e^{rt}) + \sqrt{4km}(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + \sqrt{4km}r + k = 0$$

$$r = \frac{-\sqrt{4km} \pm \sqrt{4km - 4km}}{2m} = -\frac{\sqrt{4km}}{2m} = -\sqrt{\frac{k}{m}}$$

One solution to equation (1) is $u = e^{-\sqrt{k/m}t}$. Apply the method of reduction of order to obtain the general solution: Plug in $u(t) = c(t)e^{-\sqrt{k/m}t}$ to equation (1) and solve the resulting ODE for $c(t)$.

$$m[c(t)e^{-\sqrt{k/m}t}]'' + \sqrt{4km}[c(t)e^{-\sqrt{k/m}t}]' + k[c(t)e^{-\sqrt{k/m}t}] = 0$$

Evaluate the derivatives.

$$m\left[c'(t)e^{-\sqrt{k/m}t} - c(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/m}t}\right]' + \sqrt{4km}\left[c'(t)e^{-\sqrt{k/m}t} - c(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/m}t}\right] + k[c(t)e^{-\sqrt{k/m}t}] = 0$$

$$m\left[c''(t)e^{-\sqrt{k/m}t} - \cancel{c'(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/m}t}} - \cancel{c'(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/m}t}} + \cancel{c(t)\frac{k}{m}e^{-\sqrt{k/m}t}}\right] + \sqrt{4km}\left[\cancel{c'(t)e^{-\sqrt{k/m}t}} - \cancel{c(t)\sqrt{\frac{k}{m}}e^{-\sqrt{k/m}t}}\right] + \cancel{k[c(t)e^{-\sqrt{k/m}t}]} = 0$$

$$mc''(t)e^{-\sqrt{k/m}t} = 0$$

Multiply both sides by $e^{\sqrt{k/m}t}/m$.

$$c''(t) = 0$$

Integrate both sides with respect to t.

$$c'(t) = C_1$$

Integrate both sides with respect to t once more.

$$c(t) = C_1 t + C_2$$

The general solution to equation (1) is then

$$\begin{aligned} u(t) &= c(t)e^{-\sqrt{k/m}t} \\ &= (C_1 t + C_2)e^{-\sqrt{k/m}t} \\ &= C_1 t e^{-\sqrt{k/m}t} + C_2 e^{-\sqrt{k/m}t}. \end{aligned}$$

Take a derivative of it with respect to t.

$$u'(t) = C_1 e^{-\sqrt{k/m}t} - C_1\sqrt{\frac{k}{m}}te^{-\sqrt{k/m}t} - C_2\sqrt{\frac{k}{m}}e^{-\sqrt{k/m}t}$$

Apply the initial conditions, $u(0) = u_0$ and $u'(0) = v_0$, to determine C_1 and C_2.

$$\begin{aligned} u(0) &= C_2 = u_0 \\ u'(0) &= C_1 - C_2\sqrt{\frac{k}{m}} = v_0 \end{aligned}$$

Solving this system of equations for C_1 and C_2 yields

$$C_1 = v_0 + \sqrt{\frac{k}{m}}u_0 \quad \text{and} \quad C_2 = u_0.$$

Thus, the amplitude of the critically damped system is

$$u(t) = \left(v_0 + \sqrt{\frac{k}{m}}u_0\right)te^{-\sqrt{k/m}t} + u_0 e^{-\sqrt{k/m}t}.$$

Set $u(t) = 0$ and solve for t to determine the time at which the mass passes through the equilibrium position.

$$\begin{aligned} 0 &= \left(v_0 + \sqrt{\frac{k}{m}}u_0\right)te^{-\sqrt{k/m}t} + u_0 e^{-\sqrt{k/m}t} \\ 0 &= \left(v_0 + \sqrt{\frac{k}{m}}u_0\right)t + u_0 \\ t &= -\frac{u_0}{v_0 + \sqrt{\frac{k}{m}}u_0} \end{aligned}$$

Therefore, the mass passes through the equilibrium position just once, provided that u_0 and v_0 are chosen so that this expression for t is positive. If it is not positive, then the mass never passes through the equilibrium position. For t to be positive, assuming that $u_0 > 0$, it's necessary that

$$v_0 + \sqrt{\frac{k}{m}}u_0 < 0,$$

that is,

$$v_0 < -\sqrt{\frac{k}{m}}u_0 = -\frac{\gamma}{2m}u_0.$$

If $v_0 = 0$, then the solution for $u(t)$ simplifies to

$$\begin{aligned} u(t) &= \sqrt{\frac{k}{m}} u_0 t e^{-\sqrt{k/m}t} + u_0 e^{-\sqrt{k/m}t} \\ &= u_0 e^{-\sqrt{k/m}t} \left(\sqrt{\frac{k}{m}} t + 1 \right) \\ &= u_0 \frac{\sqrt{\frac{k}{m}} t + 1}{e^{\sqrt{k/m}t}}. \end{aligned}$$

$u(t)$ could only ever be zero at $t = -\sqrt{m/k}$, but t is not negative. Now take the limit of $u(t)$ as $t \to \infty$ and apply l'Hôpital's rule.

$$\begin{aligned} \lim_{t\to\infty} u(t) &= \lim_{t\to\infty} u_0 \frac{\sqrt{\frac{k}{m}} t + 1}{e^{\sqrt{k/m}t}} \\ &\overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{t\to\infty} u_0 \frac{\sqrt{\frac{k}{m}}}{\sqrt{\frac{k}{m}} e^{\sqrt{k/m}t}} \\ &= \lim_{t\to\infty} u_0 \frac{1}{e^{\sqrt{k/m}t}} \\ &= 0 \end{aligned}$$