Problem 29

The position of a certain spring-mass system satisfies the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 0, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Find the solution of this initial value problem.

(b) Plot u versus t and u' versus t on the same axes.

(c) Plot u' versus u in the phase plane (see Problem 28). Identify several corresponding points on the curves in parts (b) and (c). What is the direction of motion on the phase plot as t increases?

Solution

Since the coefficients are constant and this ODE is homogeneous, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + \frac{1}{4}(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by $e^{rt}/4$.

$$4r^2 + r + 8 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(4)(8)}}{2(4)} = \frac{-1 \pm \sqrt{-127}}{8} = \frac{-1 \pm i\sqrt{127}}{8}$$

$$r = \left\{ \frac{-1 - i\sqrt{127}}{8}, \frac{-1 + i\sqrt{127}}{8} \right\}$$

Two solutions to the ODE are

$$u = \exp\left(\frac{-1 - i\sqrt{127}}{8}t\right) \quad \text{and} \quad u = \exp\left(\frac{-1 + i\sqrt{127}}{8}t\right).$$

By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
u(t) &= C_1 \exp\left(\frac{-1 - i\sqrt{127}}{8}t\right) + C_2 \exp\left(\frac{-1 + i\sqrt{127}}{8}t\right) \\
&= C_1 \exp\left(\frac{-t - i\sqrt{127}t}{8}\right) + C_2 \exp\left(\frac{-t + i\sqrt{127}t}{8}\right) \\
&= C_1 e^{-t/8} \exp\left(-\frac{i\sqrt{127}t}{8}\right) + C_2 e^{-t/8} \exp\left(\frac{i\sqrt{127}t}{8}\right) \\
&= C_1 e^{-t/8}\left[\cos\left(-\frac{\sqrt{127}}{8}t\right) + i\sin\left(-\frac{\sqrt{127}}{8}t\right)\right] + C_2 e^{-t/8}\left[\cos\left(\frac{\sqrt{127}}{8}t\right) + i\sin\left(\frac{\sqrt{127}}{8}t\right)\right] \\
&= C_1 e^{-t/8}\left[\cos\left(\frac{\sqrt{127}}{8}t\right) - i\sin\left(\frac{\sqrt{127}}{8}t\right)\right] + C_2 e^{-t/8}\left[\cos\left(\frac{\sqrt{127}}{8}t\right) + i\sin\left(\frac{\sqrt{127}}{8}t\right)\right] \\
&= C_1 e^{-t/8}\cos\left(\frac{\sqrt{127}}{8}t\right) - iC_1 e^{-t/8}\sin\left(\frac{\sqrt{127}}{8}t\right) + C_2 e^{-t/8}\cos\left(\frac{\sqrt{127}}{8}t\right) + iC_2 e^{-t/8}\sin\left(\frac{\sqrt{127}}{8}t\right) \\
&= (C_1 + C_2)e^{-t/8}\cos\left(\frac{\sqrt{127}}{8}t\right) + (-iC_1 + iC_2)e^{-t/8}\sin\left(\frac{\sqrt{127}}{8}t\right) \\
&= C_3 e^{-t/8}\cos\left(\frac{\sqrt{127}}{8}t\right) + C_4 e^{-t/8}\sin\left(\frac{\sqrt{127}}{8}t\right) \\
&= e^{-t/8}\left[C_3\cos\left(\frac{\sqrt{127}}{8}t\right) + C_4\sin\left(\frac{\sqrt{127}}{8}t\right)\right]
\end{aligned}$$

Differentiate it with respect to t.

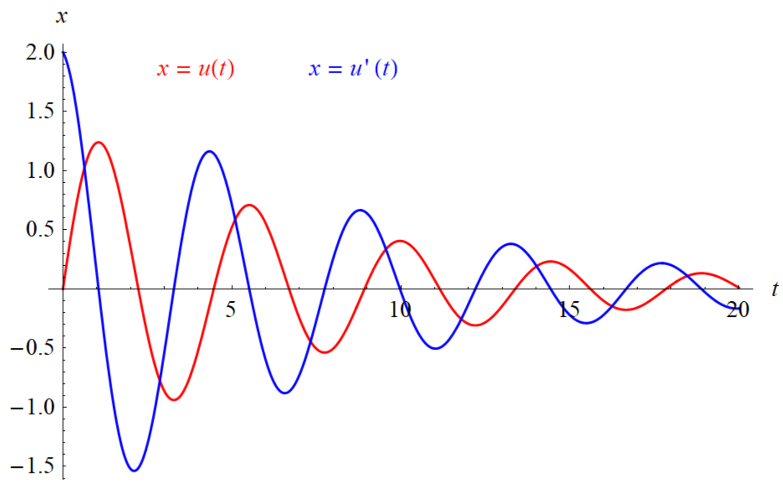
$$\begin{aligned}
u'(t) = -\frac{1}{8}e^{-t/8}\left[C_3\cos\left(\frac{\sqrt{127}}{8}t\right) + C_4\sin\left(\frac{\sqrt{127}}{8}t\right)\right] \\
+ e^{-t/8}\left[-C_3\frac{\sqrt{127}}{8}\sin\left(\frac{\sqrt{127}}{8}t\right) + C_4\frac{\sqrt{127}}{8}\cos\left(\frac{\sqrt{127}}{8}t\right)\right]
\end{aligned}$$

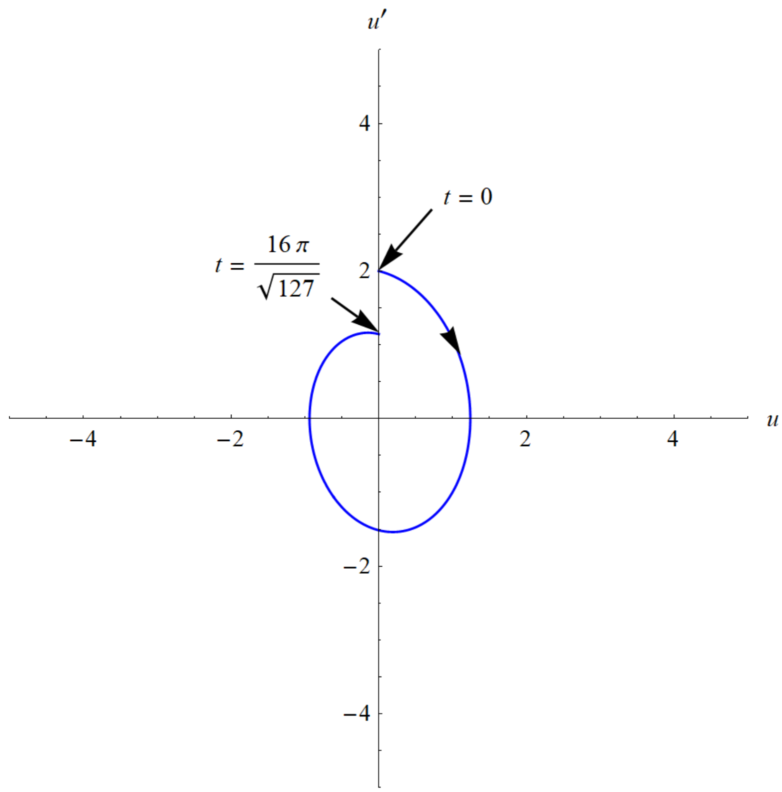
Apply the initial conditions here to determine C_3 and C_4.

$$\begin{aligned}
u(0) &= C_3 = 0 \\
u'(0) &= \frac{1}{8}(-C_3 + C_4\sqrt{127}) = 2
\end{aligned}$$

Solving this system of equations yields $C_3 = 0$ and $C_4 = 16/\sqrt{127}$. Therefore, the solution to the initial value problem is

$$u(t) = \frac{16}{\sqrt{127}}e^{-t/8}\sin\left(\frac{\sqrt{127}}{8}t\right).$$


x
2.0
1.5
1.0
0.5
−0.5
−1.0
−1.5
x = u(t)
x = u' (t)
5
10
15
20
t


u′
4
2
−2
−4
t = 0
t = 16π/√127
−4
−2
2
4
u