Problem 11

A spring is stretched 10 cm by a force of 3 N. A mass of 2 kg is hung from the spring and is also attached to a viscous damper that exerts a force of 3 N when the velocity of the mass is 5 m/s. If the mass is pulled down 5 cm below its equilibrium position and given an initial downward velocity of 10 cm/s, determine its position u at any time t. Find the quasi frequency μ and the ratio of μ to the natural frequency of the corresponding undamped motion.

Solution

Use Hooke's law to obtain the relationship between the spring force and the displacement.

$$F_s = k\Delta x$$

$$3\text{ N} = k\left(10\,\cancel{\text{cm}} \times \frac{1\text{ m}}{100\,\cancel{\text{cm}}}\right)$$

$$k = 30\,\frac{\text{N}}{\text{m}}$$

Assume that the viscous damping force is proportional to the velocity.

$$F_d \propto x'$$

Introduce a proportionality constant c to turn this into an equation we can use.

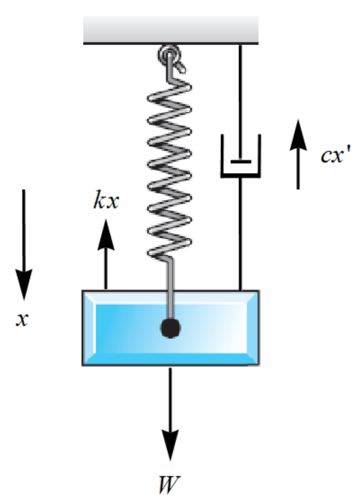
$$F_d = cx'$$

Use the fact that the damping force is 3 N when the velocity of the mass is 5 m/s to determine c.

$$3\text{ N} = c\left(5\,\frac{\text{m}}{\text{s}}\right)$$

$$c = 0.6\,\frac{\text{N}\cdot\text{s}}{\text{m}}$$

The aim now is to obtain an equation of motion. Start by drawing a free-body diagram of the mass.



Apply Newton's second law in the x-direction.

$$\sum F_x = ma_x$$

$$-cx' - kx + mg = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-cx' - kx + mg = mx''$$

$$mx'' + cx' + kx = mg$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + cx_c' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + cr + k = 0$$

$$r = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

Since $c^2 - 4mk < 0$, write the quantity under the square root as $4mk - c^2$.

$$\begin{aligned} r &= \left\{\frac{-c - i\sqrt{4mk - c^2}}{2m}, \frac{-c + i\sqrt{4mk - c^2}}{2m}\right\} \\ &= \left\{-\frac{c}{2m} - i\mu, -\frac{c}{2m} + i\mu\right\} \end{aligned}$$

Two solutions to equation (1) are $x_c = e^{(-c/2m - i\mu)t}$ and $x_c = e^{(-c/2m + i\mu)t}$. By the principle of superposition, the general solution for x_c is a linear combination of the two.

$$\begin{aligned} x_c(t) &= C_1e^{(-c/2m - i\mu)t} + C_2e^{(-c/2m + i\mu)t} \\ &= C_1e^{-ct/2m - i\mu t} + C_2e^{-ct/2m + i\mu t} \\ &= C_1e^{-ct/2m}e^{-i\mu t} + C_2e^{-ct/2m}e^{i\mu t} \\ &= C_1e^{-ct/2m}[\cos(-\mu t) + i\sin(-\mu t)] + C_2e^{-ct/2m}[\cos(\mu t) + i\sin(\mu t)] \\ &= C_1e^{-ct/2m}[\cos(\mu t) - i\sin(\mu t)] + C_2e^{-ct/2m}[\cos(\mu t) + i\sin(\mu t)] \\ &= C_1e^{-ct/2m}\cos\mu t - iC_1e^{-ct/2m}\sin\mu t + C_2e^{-ct/2m}\cos\mu t + iC_2e^{-ct/2m}\sin\mu t \\ &= (C_1 + C_2)e^{-ct/2m}\cos\mu t + (-iC_1 + iC_2)e^{-ct/2m}\sin\mu t \\ &= C_3e^{-ct/2m}\cos\mu t + C_4e^{-ct/2m}\sin\mu t \end{aligned}$$

On the other hand, the particular solution satisfies

$$mx_p'' + cx_p' + kx_p = mg.$$

Because the inhomogeneous term is a constant, the particular solution is a constant as well: $x_p(t) = A$. Substitute this into the equation to determine A.

$$m(A)'' + c(A)' + k(A) = mg$$

$$kA = mg$$
$$A = \frac{mg}{k}$$

So then $x_p(t) = mg/k$, which means that the general solution for x is

$$x(t) = C_3 e^{-ct/2m}\cos\mu t + C_4 e^{-ct/2m}\sin\mu t + \frac{mg}{k}.$$

Take a derivative of it with respect to t.

$$x'(t) = -C_3\frac{c}{2m}e^{-ct/2m}\cos\mu t - C_3\mu e^{-ct/2m}\sin\mu t - C_4\frac{c}{2m}e^{-ct/2m}\sin\mu t + C_4\mu e^{-ct/2m}\cos\mu t$$

Now we will use initial conditions to find C_3 and C_4. To determine the first initial condition, $x(0)$, we need to know the equilibrium height of the mass. Use Hooke's law once again.

$$mg = kx_{\text{eq}}$$

$$x_{\text{eq}} = \frac{mg}{k} = \frac{(2\text{ kg})\left(9.81\ \frac{\text{m}}{\text{s}^2}\right)}{30\ \frac{\text{N}}{\text{m}}} = 0.654\text{ m} = 65.4\text{ cm}$$

Consequently, the initial conditions are

$$x(0) = x_{\text{eq}} + 5\ \cancel{\text{cm}} \times \frac{1\text{ m}}{100\ \cancel{\text{cm}}} = \frac{mg}{k} + 0.05\text{ m}$$
$$x'(0) = 10\ \frac{\cancel{\text{cm}}}{\text{s}} \times \frac{1\text{ m}}{100\ \cancel{\text{cm}}} = 0.1\ \frac{\text{m}}{\text{s}}.$$

Apply them to get

$$x(0) = C_3 + \frac{mg}{k} = \frac{mg}{k} + 0.05$$
$$x'(0) = -C_3\frac{c}{2m} + C_4\mu = 0.1.$$

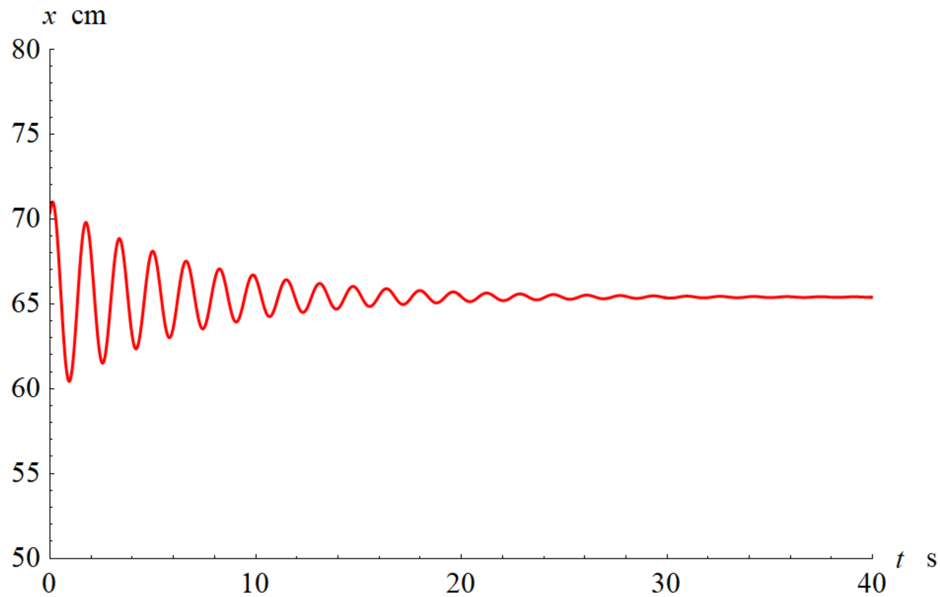
Solving this system of equations yields

$$C_3 = 0.05 \quad \text{and} \quad C_4 = \frac{1}{\mu}\left(0.1 + \frac{0.05c}{2m}\right).$$

Therefore,

$$\begin{aligned} x(t) &= 0.05e^{-ct/2m}\cos\mu t + \frac{1}{\mu}\left(0.1 + \frac{0.05c}{2m}\right)e^{-ct/2m}\sin\mu t + \frac{mg}{k} \\ &= 0.05e^{-0.15t}\cos\frac{\sqrt{4mk - c^2}}{2m}t + \frac{2m}{\sqrt{4mk - c^2}}\left(0.1 + \frac{0.05c}{2m}\right)e^{-0.15t}\sin\frac{\sqrt{4mk - c^2}}{2m}t + \frac{mg}{k}. \end{aligned}$$

As $x(t)$ is in meters, multiply the result by 100 to get it in centimeters.



The quasi angular frequency is

$$\mu = \frac{\sqrt{4mk - c^2}}{2m} \approx 3.87 \, \frac{\text{rad}}{\text{s}},$$

and the ratio of μ to the natural frequency of the corresponding undamped motion is

$$\frac{\mu}{\omega} = \frac{\frac{\sqrt{4mk-c^2}}{2m}}{\sqrt{\frac{k}{m}}} \approx 0.99925.$$