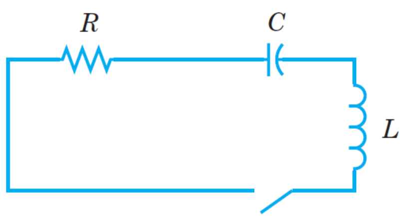
Problem 12

A series circuit has a capacitor of 10^{-5} F, a resistor of 3×10^2 Ω, and an inductor of 0.2 H. The initial charge on the capacitor is 10^{-6} C and there is no initial current. Find the charge Q on the capacitor at any time t.

Solution

Start by drawing an RLC series circuit.



Assume that the circuit is closed at $t = 0$. Apply Faraday's law to obtain the governing ODE for the current.

$$\sum V = -L\frac{di}{dt}$$

The only potential drops occur over the resistor and the capacitor.

$$iR + \frac{q}{C} = -L\frac{di}{dt}$$

Write $i = dq/dt = q'$.

$$Rq' + \frac{q}{C} = -Lq''$$

Bring Lq'' to the left side and divide both sides by L.

$$q'' + \frac{R}{L}q' + \frac{1}{LC}q = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $q = e^{rt}$.

$$q = e^{rt} \quad \rightarrow \quad q' = re^{rt} \quad \rightarrow \quad q'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2e^{rt} + \frac{R}{L}(re^{rt}) + \frac{1}{LC}(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + \frac{R}{L}r + \frac{1}{LC} = 0$$

$$r = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - \frac{4}{LC}}}{2} = \frac{-1500 \pm \sqrt{250\,000}}{2} = \frac{-1500 \pm 500}{2} = -750 \pm 250$$

$$r = \{-1000, -500\}$$

Two solutions to equation (1) are then $q = e^{-1000t}$ and $q = e^{-500t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$q(t) = C_1e^{-1000t} + C_2e^{-500t}$$

Take a derivative of it with respect to t.

$$q'(t) = -1000C_1e^{-1000t} - 500C_2e^{-500t}$$

Now apply the initial conditions, $q(0) = 10^{-6}$ and $q'(0) = 0$, to determine C_1 and C_2.

$$q(0) = C_1 + C_2 = 10^{-6}$$
$$q'(0) = -1000C_1 - 500C_2 = 0$$

Solving this system of equations yields $C_1 = -10^{-6}$ and $C_2 = 2 \times 10^{-6}$. Therefore,

$$q(t) = -10^{-6}e^{-1000t} + 2 \times 10^{-6}e^{-500t}.$$

Note that the charge $q(t)$ is in Coulombs (C).