Problem 16

Show that $A\cos\omega_0 t + B\sin\omega_0 t$ can be written in the form $r\sin(\omega_0 t - \theta)$. Determine r and θ in terms of A and B. If $R\cos(\omega_0 t - \delta) = r\sin(\omega_0 t - \theta)$, determine the relationship among R, r, δ, and θ.

Solution

$A\cos\omega_0 t + B\sin\omega_0 t$ can be written in the form $r\sin(\omega_0 t - \theta)$, provided that A and B satisfy the following system of equations.

$$-r\sin\theta = A \tag{1}$$
$$r\cos\theta = B \tag{2}$$

Assuming that they do, then

$$\begin{aligned} A\cos\omega_0 t + B\sin\omega_0 t &= -r\sin\theta\cos\omega_0 t + r\cos\theta\sin\omega_0 t \\ &= r(\sin\omega_0 t\cos\theta - \cos\omega_0 t\sin\theta) \\ &= r\sin(\omega_0 t - \theta). \end{aligned}$$

Solve equations (1) and (2) for r first. Square both sides of each equation

$$r^2\sin^2\theta = A^2$$
$$r^2\cos^2\theta = B^2$$

and then add the respective sides.

$$r^2\sin^2\theta + r^2\cos^2\theta = A^2 + B^2$$

$$r^2 = A^2 + B^2$$

$$\boxed{r = \sqrt{A^2 + B^2}}$$

Divide the respective sides of equation (1) by those of equation (2) to find δ.

$$\frac{-r\sin\theta}{r\cos\theta} = \frac{A}{B}$$

$$-\tan\theta = \frac{A}{B}$$

$$\tan\theta = -\frac{A}{B}$$

$$\boxed{\theta = \tan^{-1}\left(-\frac{A}{B}\right)}$$

Suppose that $R\cos(\omega_0 t - \delta) = r\sin(\omega_0 t - \theta)$. Use the trigonometric identity, $\cos x = \sin(x + \pi/2)$, so that both sides are in terms of sine.

$$R\sin\left(\omega_0 t - \delta + \frac{\pi}{2}\right) = r\sin(\omega_0 t - \theta)$$

From this equation, we have

$$R = r \qquad\qquad \begin{aligned} \omega_0 t - \delta + \frac{\pi}{2} + 2\pi n &= \omega_0 t - \theta \\ -\delta + \frac{\pi + 4\pi n}{2} &= -\theta \\ \delta &= \theta + \frac{(1+4n)\pi}{2}, \end{aligned}$$

where $n = 0, \pm 1, \pm 2, \ldots$.