Problem 21

Logarithmic Decrement.

(a) For the damped oscillation described by Eq. (26), show that the time between successive maxima is $T_d = 2\pi/\mu$.

(b) Show that the ratio of the displacements at two successive maxima is given by $\exp(\gamma T_d/2m)$. Observe that this ratio does not depend on which pair of maxima is chosen. The natural logarithm of this ratio is called the logarithmic decrement and is denoted by Δ.

(c) Show that $\Delta = \pi\gamma/m\mu$. Since m, μ, and Δ are quantities that can be measured easily for a mechanical system, this result provides a convenient and practical method for determining the damping constant of the system, which is more difficult to measure directly. In particular, for the motion of a vibrating mass in a viscous fluid, the damping constant depends on the viscosity of the fluid; for simple geometric shapes the form of this dependence is known, and the preceding relation allows the experimental determination of the viscosity. This is one of the most accurate ways of determining the viscosity of a gas at high pressure.

Solution

Part (a)

Eq. (26) is

$$u = Re^{-\gamma t/2m}\cos(\mu t - \delta). \tag{26}$$

It gives the amplitude for a system governed by $mu'' + \gamma u' + ku = 0$ with the condition that $\gamma^2 - 4km < 0$. The maxima occur when the cosine is 1.

$$\cos(\mu t - \delta) = 1$$

$$\mu t_n - \delta = 2\pi n, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$t_n = \frac{2\pi n + \delta}{\mu}$$

The time between successive maxima is therefore

$$\begin{aligned} T_d &= t_{n+1} - t_n \\ &= \frac{2\pi(n+1) + \delta}{\mu} - \frac{2\pi n + \delta}{\mu} \\ &= \frac{\cancel{2\pi n} + 2\pi + \cancel{\delta} - \cancel{2\pi n} - \cancel{\delta}}{\mu} \\ &= \frac{2\pi}{\mu}. \end{aligned}$$

Part (b)

The maxima are

$$u_n = Re^{-\gamma t_n/2m}, \quad n = 0, \pm 1, \pm 2, \ldots,$$

so the ratio of two successive maxima is

$$\begin{aligned}
\frac{u_n}{u_{n+1}} &= \frac{Re^{-\gamma t_n/2m}}{Re^{-\gamma t_{n+1}/2m}} \\
&= \exp\left(-\frac{\gamma t_n}{2m} + \frac{\gamma t_{n+1}}{2m}\right) \\
&= \exp\left[\frac{\gamma}{2m}(t_{n+1} - t_n)\right] \\
&= \exp\left(\frac{\gamma T_d}{2m}\right).
\end{aligned}$$

Part (c)

Take the natural logarithm of both sides.

$$\begin{aligned}
\ln \frac{u_n}{u_{n+1}} &= \ln \exp\left(\frac{\gamma T_d}{2m}\right) \\
\Delta &= \frac{\gamma T_d}{2m} \\
&= \frac{\gamma}{2m}\frac{2\pi}{\mu} \\
&= \frac{\pi\gamma}{m\mu}
\end{aligned}$$