Problem 24

The position of a certain spring-mass system satisfies the initial value problem

$$\frac{3}{2}u'' + ku = 0, \qquad u(0) = 2, \quad u'(0) = v.$$

If the period and amplitude of the resulting motion are observed to be π and 3, respectively, determine the values of k and v.

Solution

Multiply both sides of the ODE by 2/3.

$$u'' + \frac{2k}{3}u = 0$$

The general solution is

$$u(t) = C_1 \cos\sqrt{\frac{2k}{3}}t + C_2 \sin\sqrt{\frac{2k}{3}}t.$$

Take a derivative of it with respect to t.

$$u'(t) = -C_1\sqrt{\frac{2k}{3}}\sin\sqrt{\frac{2k}{3}}t + C_2\sqrt{\frac{2k}{3}}\cos\sqrt{\frac{2k}{3}}t$$

Apply the initial conditions to determine C_1 and C_2.

$$u(0) = C_1 = 2$$
$$u'(0) = C_2\sqrt{\frac{2k}{3}} = v \quad \rightarrow \quad C_2 = v\sqrt{\frac{3}{2k}}$$

The solution to the initial value problem is then

$$u(t) = 2\cos\sqrt{\frac{2k}{3}}t + v\sqrt{\frac{3}{2k}}\sin\sqrt{\frac{2k}{3}}t.$$

Introduce an amplitude R and phase δ to combine the two sinusoidal terms into one.

$$\begin{aligned} u(t) &= R\cos\delta\cos\sqrt{\frac{2k}{3}}t + R\sin\delta\sin\sqrt{\frac{2k}{3}}t \\ &= R\cos\left(\sqrt{\frac{2k}{3}}t - \delta\right) \end{aligned}$$

R and δ satisfy the following system of equations.

$$R\cos\delta = 2$$
$$R\sin\delta = v\sqrt{\frac{3}{2k}}$$

Square both sides of each equation

$$R^2\cos^2\delta = 4$$
$$R^2\sin^2\delta = v^2\frac{3}{2k}$$

and then add the respective sides to determine R.

$$R^2\cos^2\delta + R^2\sin^2\delta = 4 + v^2\frac{3}{2k}$$

$$R^2 = 4 + v^2\frac{3}{2k}$$

$$R = \sqrt{4 + \frac{3v^2}{2k}}$$

On the other hand, the period of the motion is

$$T = \frac{2\pi}{\sqrt{\frac{2k}{3}}} = 2\pi\sqrt{\frac{3}{2k}}.$$

Use the fact that the period and amplitude are π and 3, respectively, to determine the values of k and v.

$$T = 2\pi\sqrt{\frac{3}{2k}} = \pi$$

$$R = \sqrt{4 + \frac{3v^2}{2k}} = 3$$

Solving this system of equations yields $k = 6$ and $v = \pm 2\sqrt{5}$.