Problem 25

Consider the initial value problem

$$u'' + \gamma u' + u = 0, \qquad u(0) = 2, \quad u'(0) = 0.$$

We wish to explore how long a time interval is required for the solution to become "negligible" and how this interval depends on the damping coefficient γ. To be more precise, let us seek the time τ such that $|u(t)| < 0.01$ for all $t > \tau$. Note that critical damping for this problem occurs for $\gamma = 2$.

(a) Let $\gamma = 0.25$ and determine τ, or at least estimate it fairly accurately from a plot of the solution.

(b) Repeat part (a) for several other values of γ in the interval $0 < \gamma < 1.5$. Note that τ steadily decreases as γ increases for γ in this range.

(c) Create a graph of τ versus γ by plotting the pairs of values found in parts (a) and (b). Is the graph a smooth curve?

(d) Repeat part (b) for values of γ between 1.5 and 2. Show that τ continues to decrease until γ reaches a certain critical value γ_0, after which τ increases. Find γ_0 and the corresponding minimum value of τ to two decimal places.

(e) Another way to proceed is to write the solution of the initial value problem in the form (26). Neglect the cosine factor and consider only the exponential factor and the amplitude R. Then find an expression for τ as a function of γ. Compare the approximate results obtained in this way with the values determined in parts (a), (b), and (d).

Solution

Begin by solving the initial value problem. Since the coefficients are constant and this ODE is homogeneous, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \to \quad u = re^{rt} \quad \to \quad u'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2 e^{rt} + \gamma(re^{rt}) + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + \gamma r + 1 = 0$$

Since $0 < \gamma < 2$ in this problem, r will have an imaginary component.

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2} = \frac{-\gamma \pm i\sqrt{4 - \gamma^2}}{2} = -\frac{\gamma}{2} \pm i\mu$$

Two solutions to the ODE are $u = e^{(-\gamma/2 - i\mu)t}$ and $u = e^{(-\gamma/2 + i\mu)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
u(t) &= C_1e^{(-\gamma/2-i\mu)t} + C_2e^{(-\gamma/2+i\mu)t} \\
&= C_1e^{-\gamma t/2-i\mu t} + C_2e^{-\gamma t/2+i\mu t} \\
&= C_1e^{-\gamma t/2}e^{-i\mu t} + C_2e^{-\gamma t/2}e^{i\mu t} \\
&= C_1e^{-\gamma t/2}[\cos(-\mu t) + i\sin(-\mu t)] + C_2e^{-\gamma t/2}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1e^{-\gamma t/2}[\cos(\mu t) - i\sin(\mu t)] + C_2e^{-\gamma t/2}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1e^{-\gamma t/2}\cos\mu t - iC_1e^{-\gamma t/2}\sin\mu t + C_2e^{-\gamma t/2}\cos\mu t + iC_2e^{-\gamma t/2}\sin\mu t \\
&= (C_1+C_2)e^{-\gamma t/2}\cos\mu t + (-iC_1+iC_2)e^{-\gamma t/2}\sin\mu t \\
&= C_3e^{-\gamma t/2}\cos\mu t + C_4e^{-\gamma t/2}\sin\mu t \\
&= e^{-\gamma t/2}(C_3\cos\mu t + C_4\sin\mu t)
\end{aligned}$$

Differentiate it with respect to t.

$$u'(t) = -\frac{\gamma}{2}e^{-\gamma t/2}(C_3\cos\mu t + C_4\sin\mu t) + e^{-\gamma t/2}(-C_3\mu\sin\mu t + C_4\mu\cos\mu t)$$

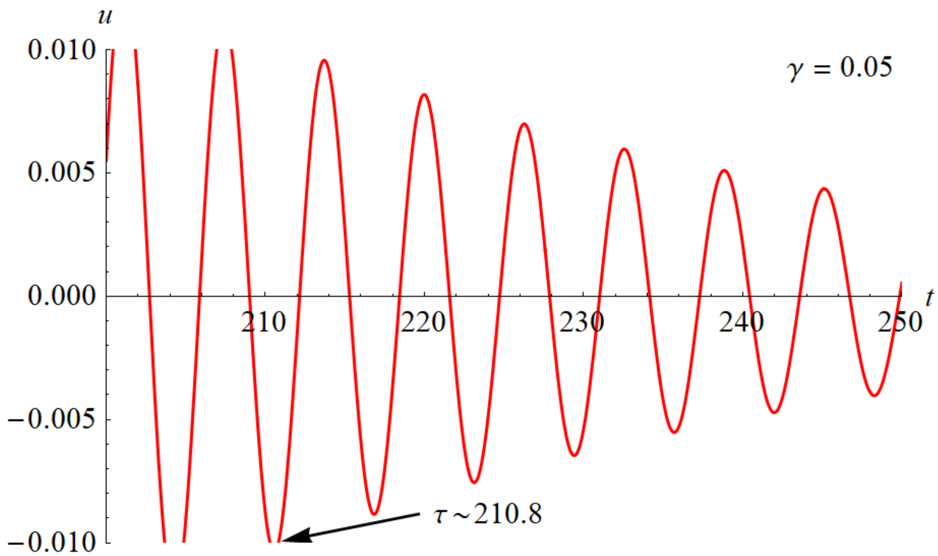
Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned}
u(0) &= C_3 = 2 \\
u'(0) &= -\frac{\gamma}{2}C_3 + \mu C_4 = 0
\end{aligned}$$

Solving this system of equations yields $C_3 = 2$ and $C_4 = \gamma/\mu$. Therefore, the solution to the initial value problem is

$$\begin{aligned}
u(t) &= e^{-\gamma t/2}\left(2\cos\mu t + \frac{\gamma}{\mu}\sin\mu t\right) \\
&= e^{-\gamma t/2}\left(2\cos\frac{\sqrt{4-\gamma^2}}{2}t + \frac{2\gamma}{\sqrt{4-\gamma^2}}\sin\frac{\sqrt{4-\gamma^2}}{2}t\right).
\end{aligned}$$

We will now plot the solution, starting with $\gamma = 0.05$ and incrementing by 0.05 until $\gamma = 1.95$ is reached. The time τ at which the amplitude is less than 0.01 for all $t > \tau$ will be found in each case.


u
0.010
0.005
0.000
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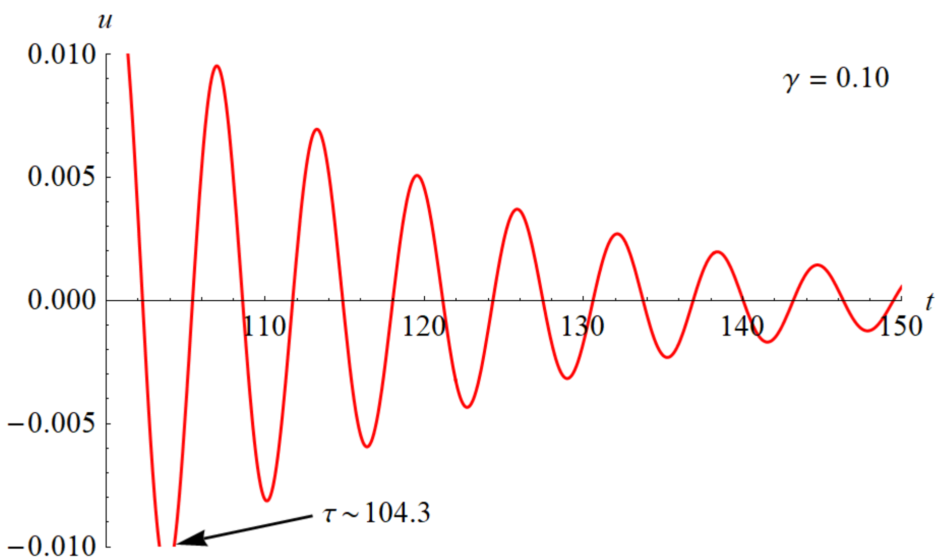
210
220
230
240
250
t
γ = 0.05
τ ~ 210.8


u
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0.000
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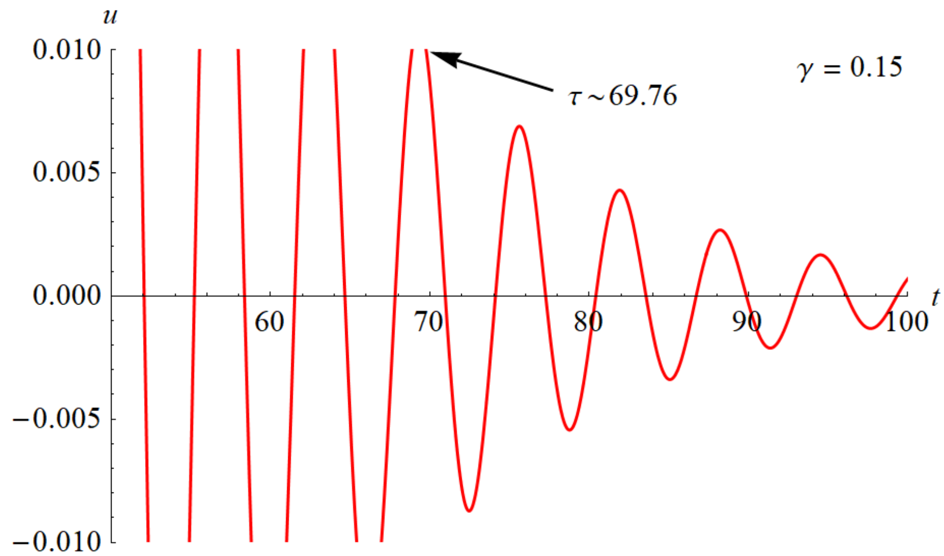
−0.010
110
120
130
140
150
t
γ = 0.10
τ ~ 104.3


u
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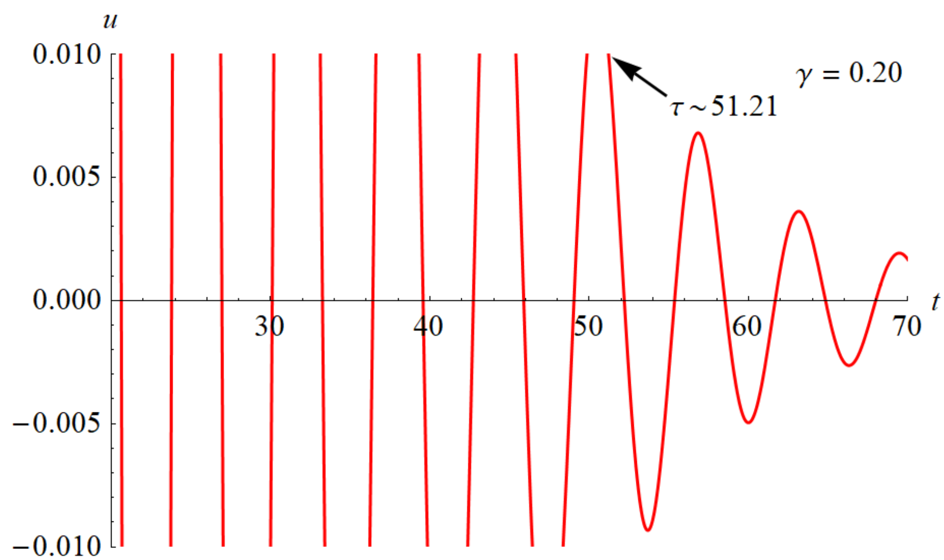
60
70
80
90
100
t
τ ~ 69.76
γ = 0.15


u
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0.000
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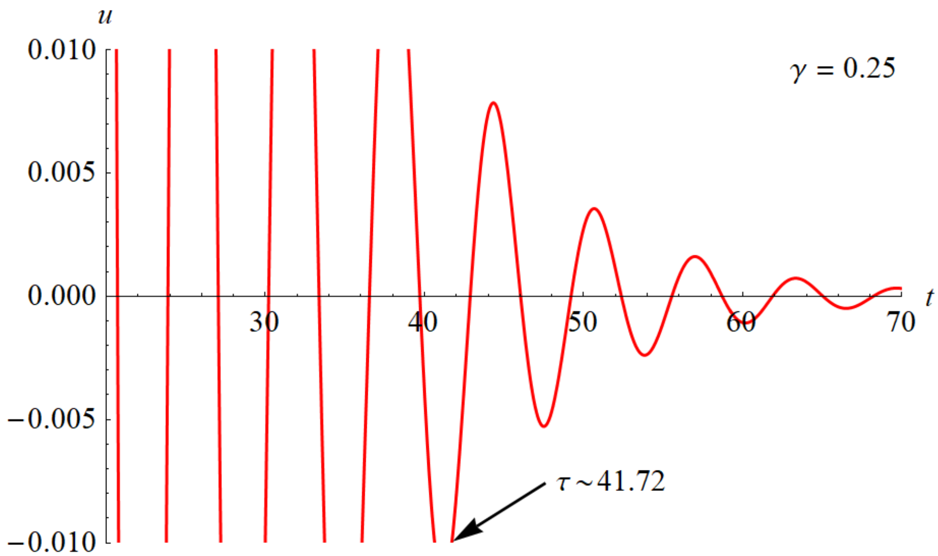
−0.010
30
40
50
60
70
t
τ ~ 51.21
γ = 0.20


u
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0.000
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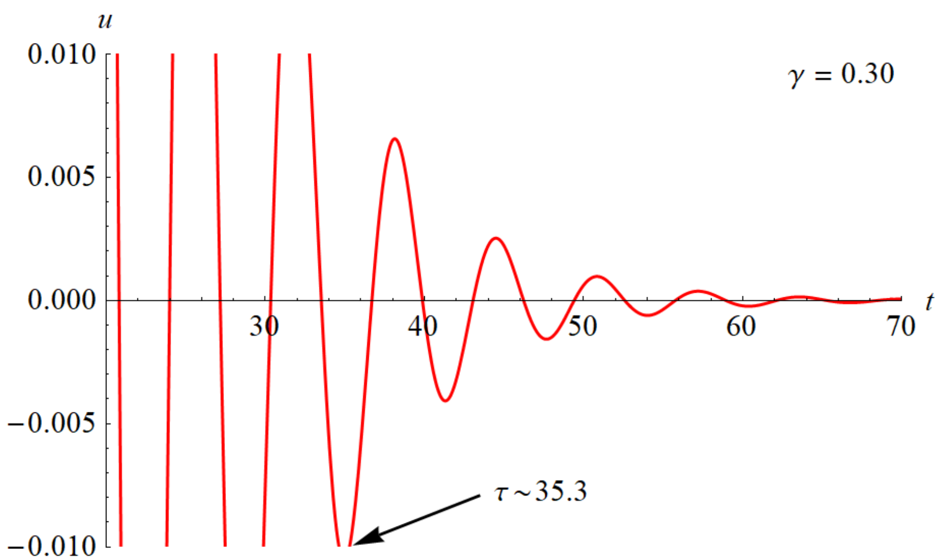
30
40
50
60
70
t
γ = 0.25
τ ∼ 41.72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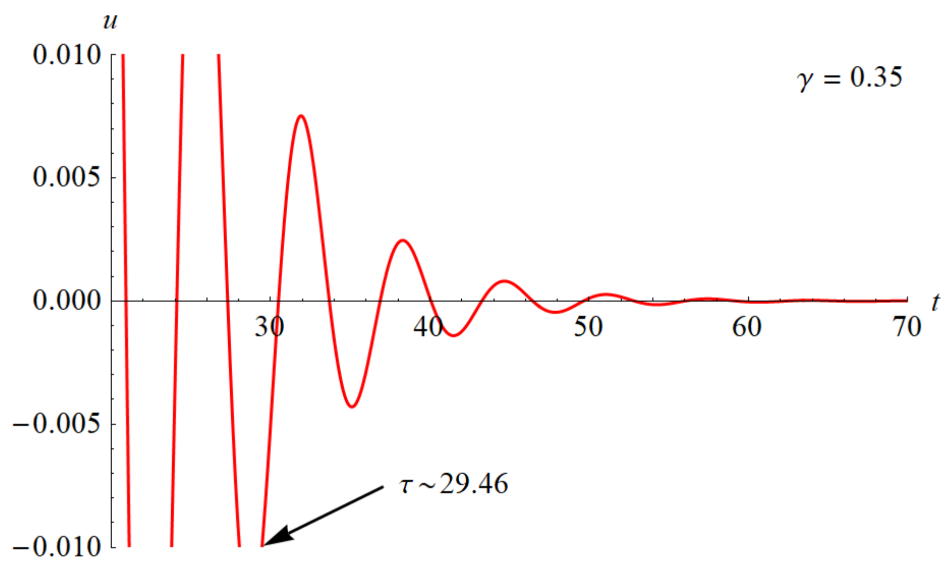
30
40
50
60
70
t
γ = 0.30
τ ∼ 35.3


u
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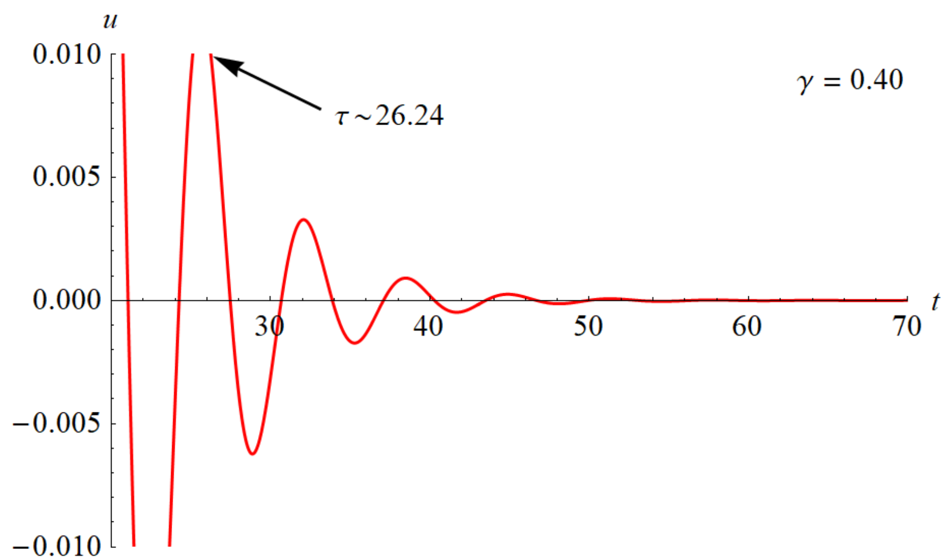
30
40
50
60
70
t
γ = 0.35
τ ~ 29.46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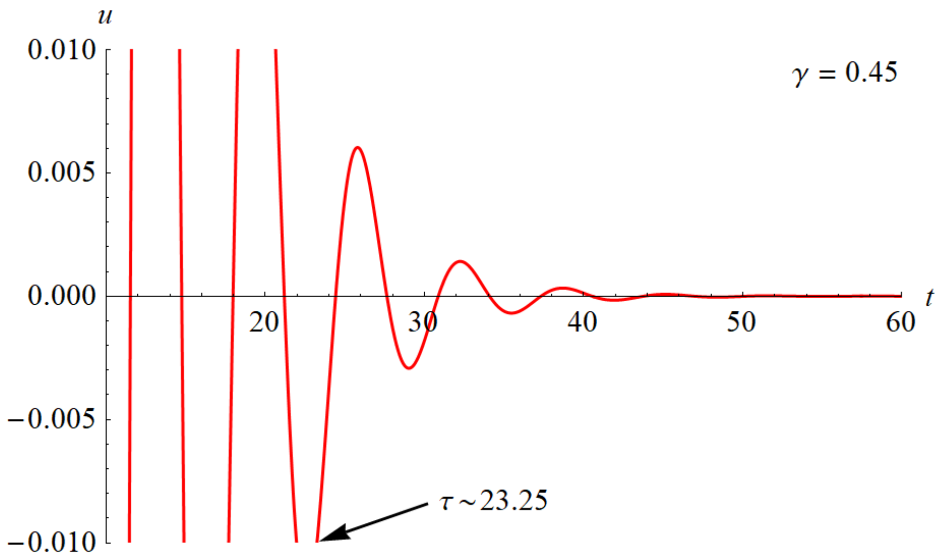
50
60
70
t
γ = 0.40
τ ~ 26.24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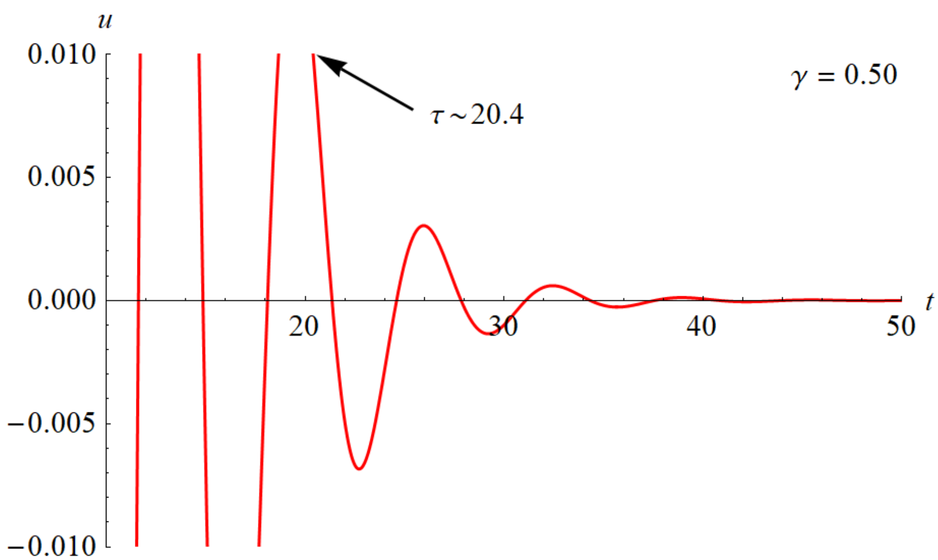
20
30
40
50
60
t
γ = 0.45
τ ~ 23.25


u
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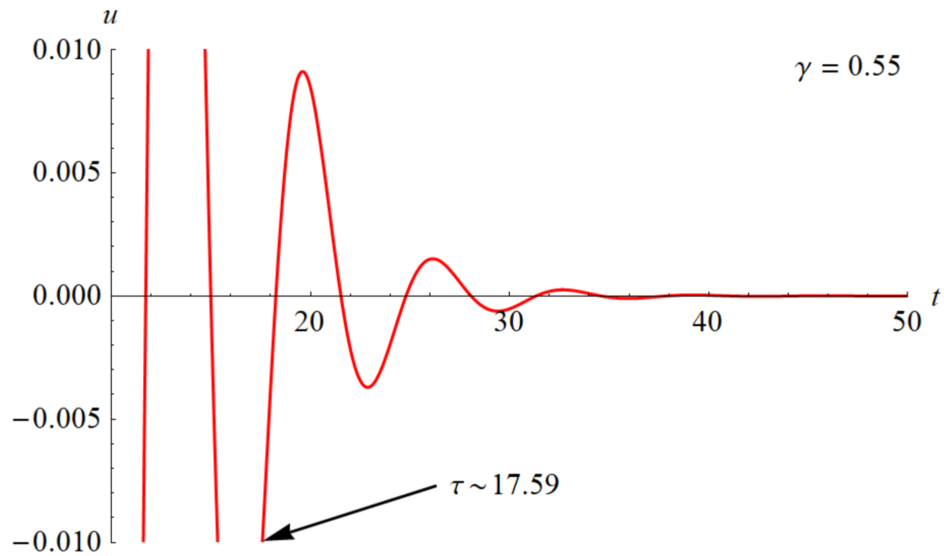
0.000
−0.005
−0.010
20
30
40
50
t
γ = 0.50
τ ~ 20.4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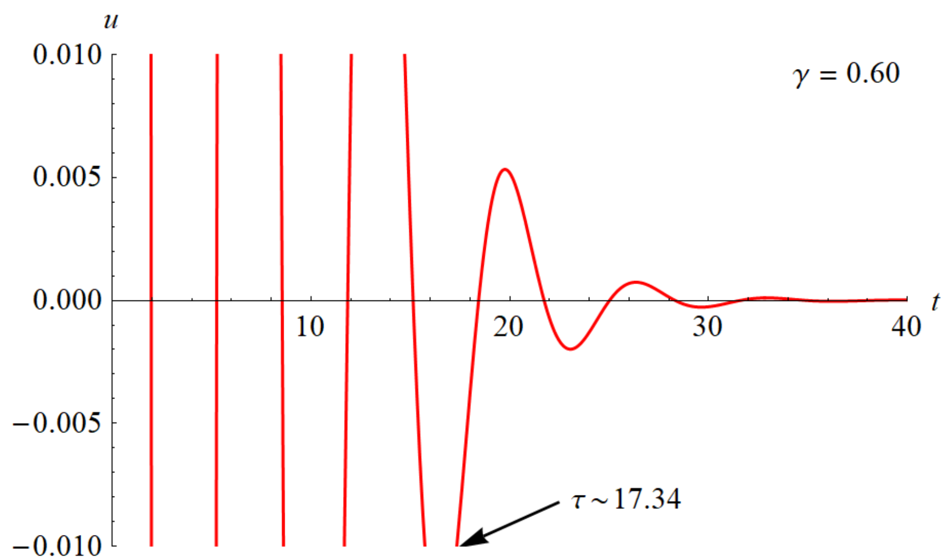
20
30
40
50
t
γ = 0.55
τ ~ 17.59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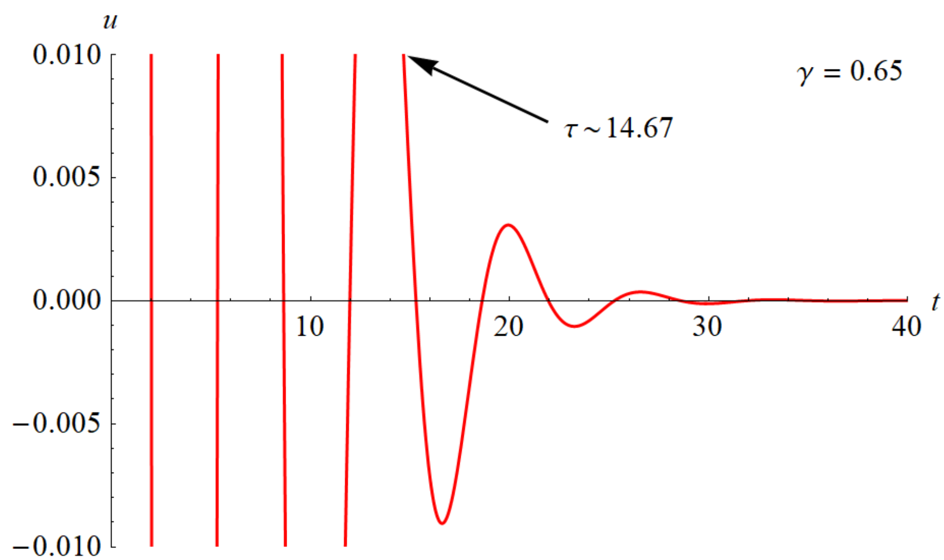
10
20
30
40
t
γ = 0.60
τ ~ 17.34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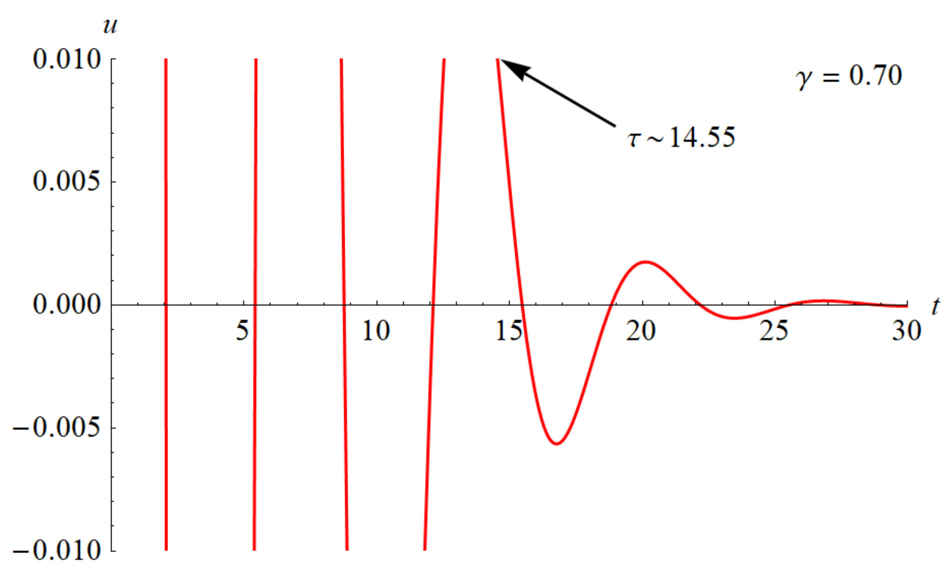
10
20
30
40
t
γ = 0.65
τ ~ 14.67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0.000
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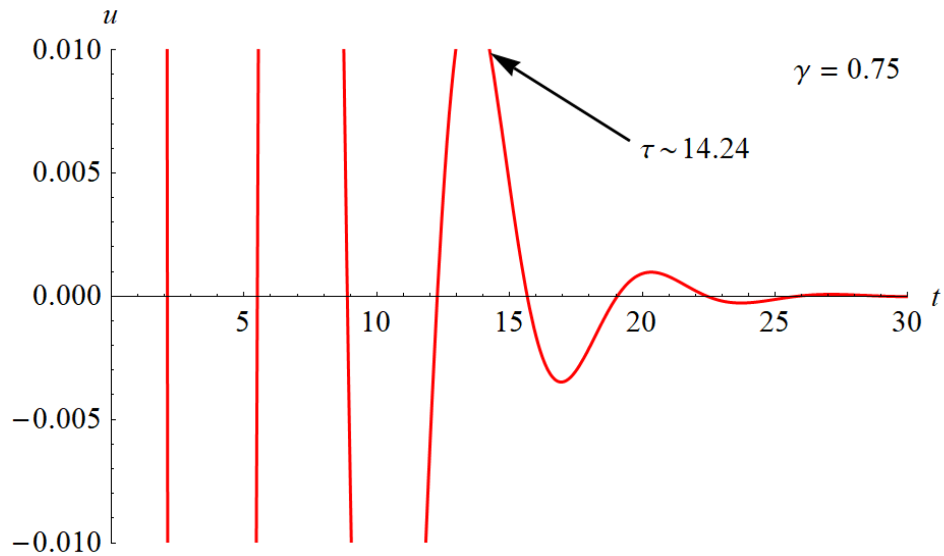
−0.010
5
10
15
20
25
30
t
γ = 0.70
τ ~ 14.55


u
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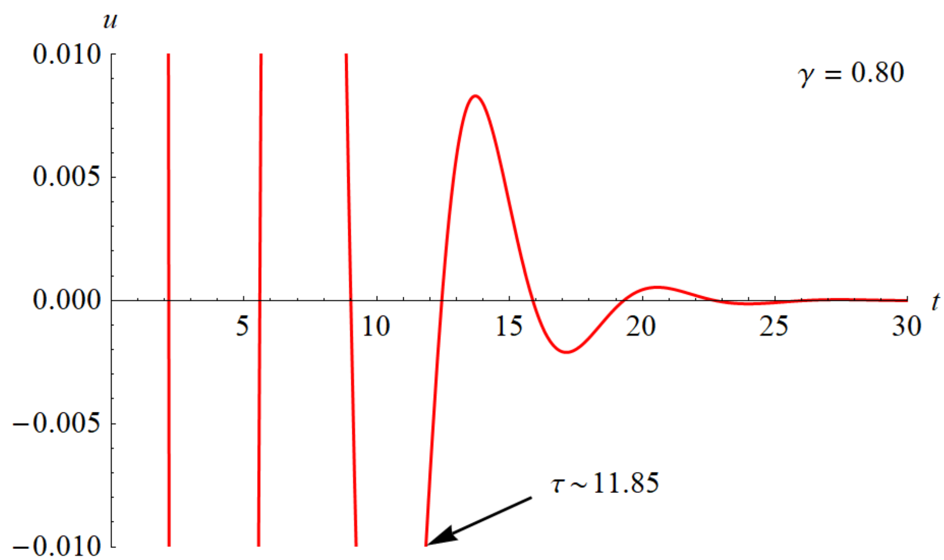
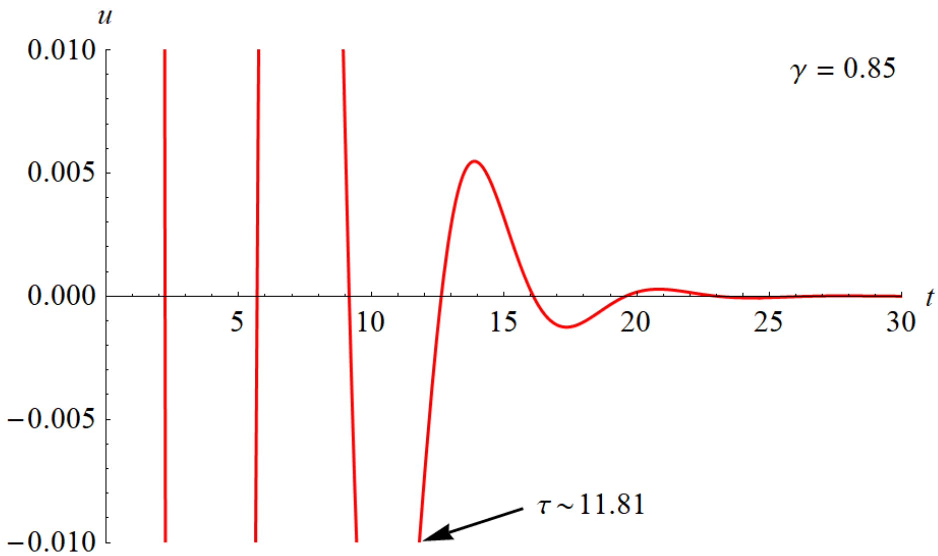
−0.010
5
10
15
20
25
30
t
γ = 0.75
τ ∼ 14.24


u
0.010
0.005
0.000
−0.005
−0.010
5
10
15
20
25
30
t
γ = 0.80
τ ∼ 11.85


u
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0.000
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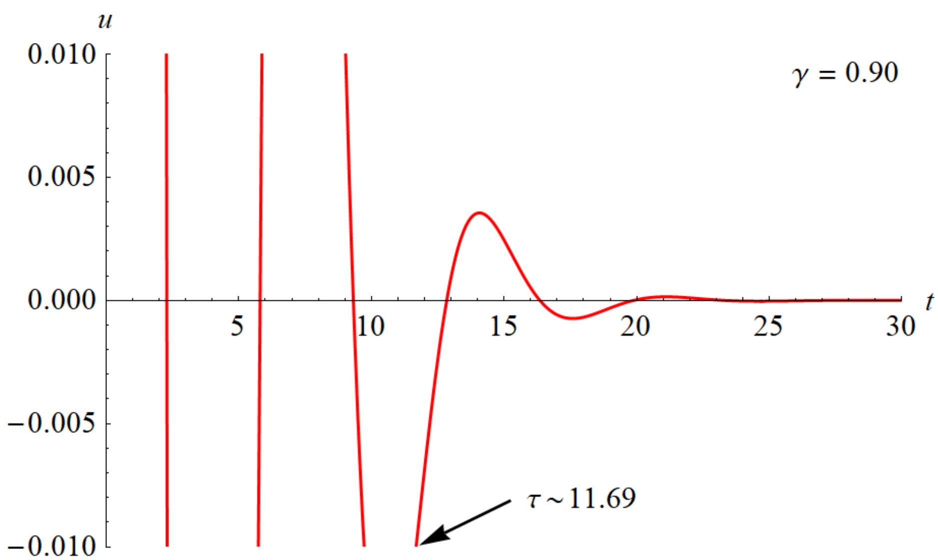
5
10
15
20
25
30
t
γ = 0.85
τ ~ 11.81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0.000
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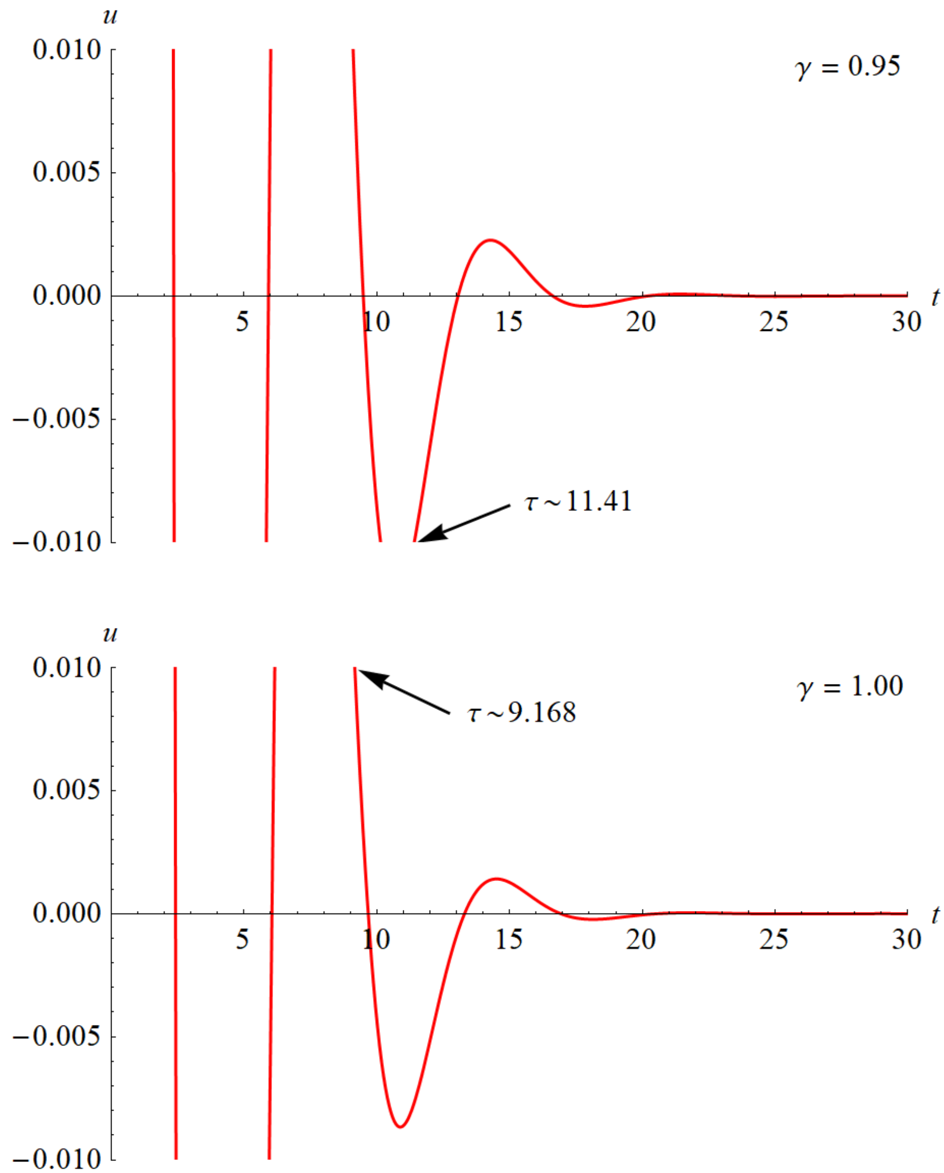
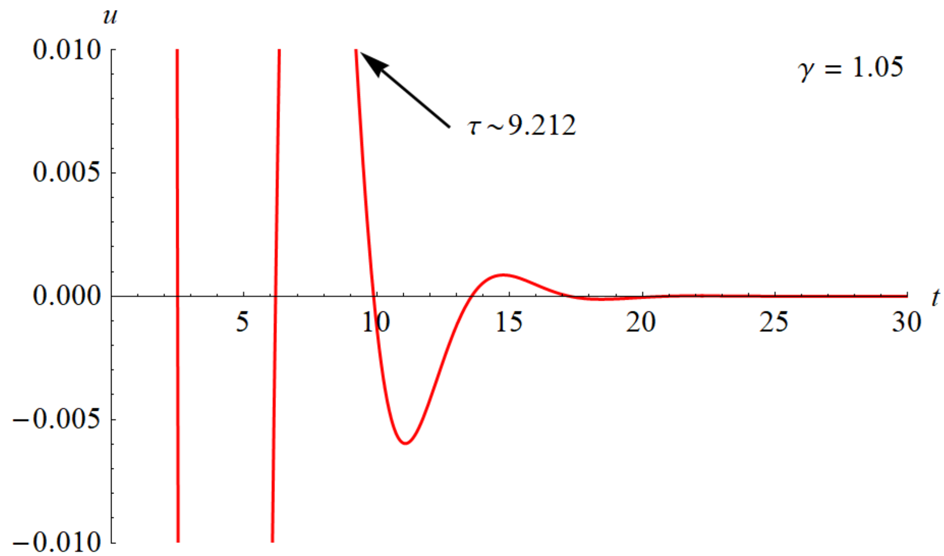
−0.010
5
10
15
20
25
30
t
γ = 0.90
τ ~ 11.69



u
0.010
0.005
0.000
−0.005
−0.010
5
10
15
20
25
30
t
γ = 0.95
τ ~ 11.41
u
0.010
0.005
0.000
−0.005
−0.010
5
10
15
20
25
30
t
γ = 1.00
τ ~ 9.168


u
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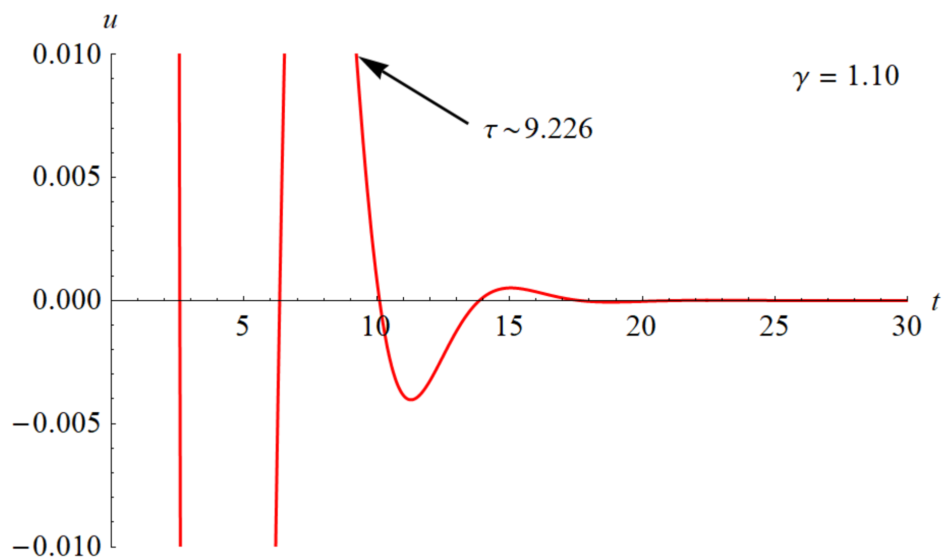
10
15
20
25
30
t
τ ~ 9.212
γ = 1.05


u
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0.000
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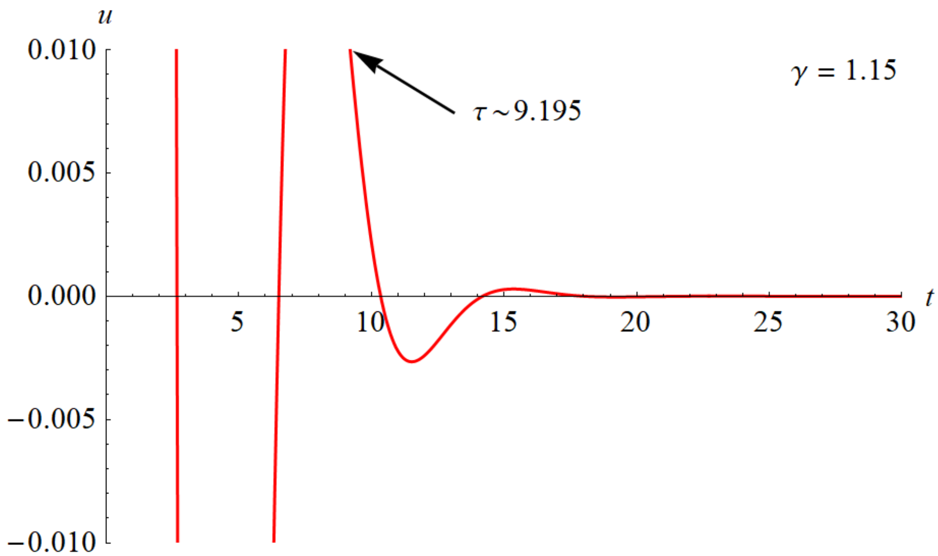
−0.010
5
10
15
20
25
30
t
τ ~ 9.226
γ = 1.10


u
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0.000
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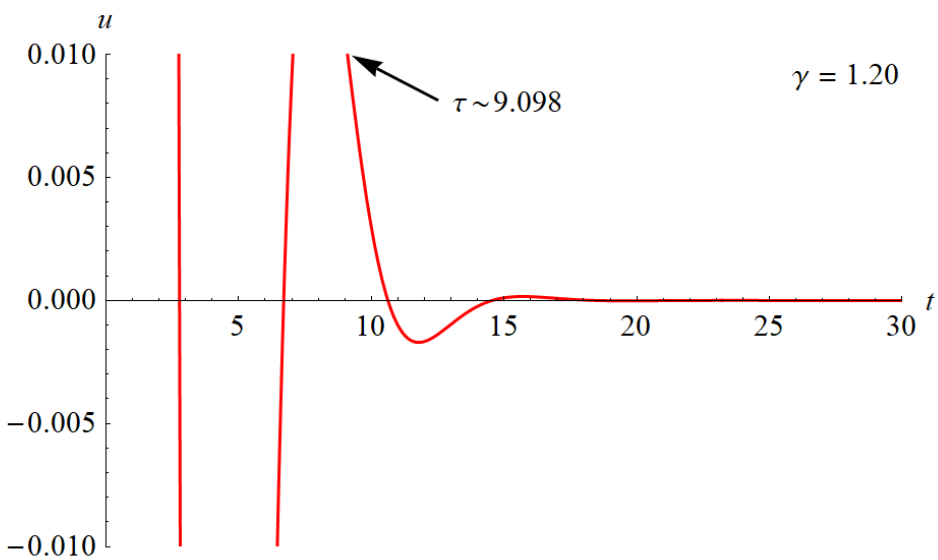
−0.010
5
10
15
20
25
30
t
τ ~ 9.195
γ = 1.15


u
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0.005
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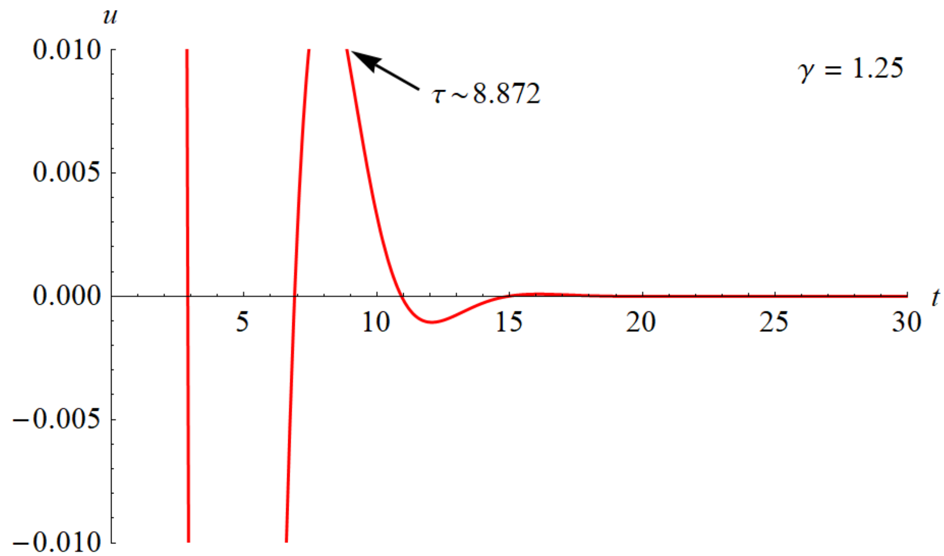
−0.010
5
10
15
20
25
30
t
τ ~ 9.098
γ = 1.20


u
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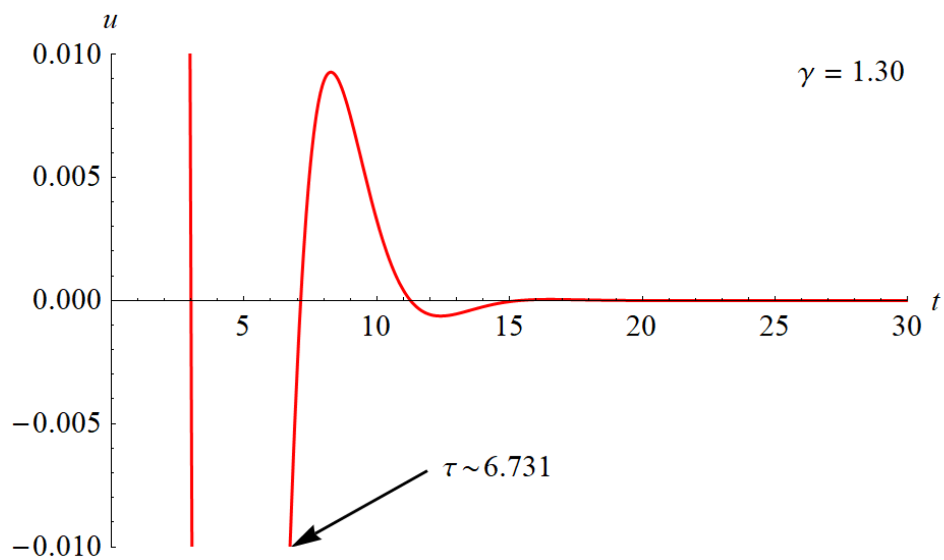
−0.010
5
10
15
20
25
30
t
τ ~ 8.872
γ = 1.25


u
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0.005
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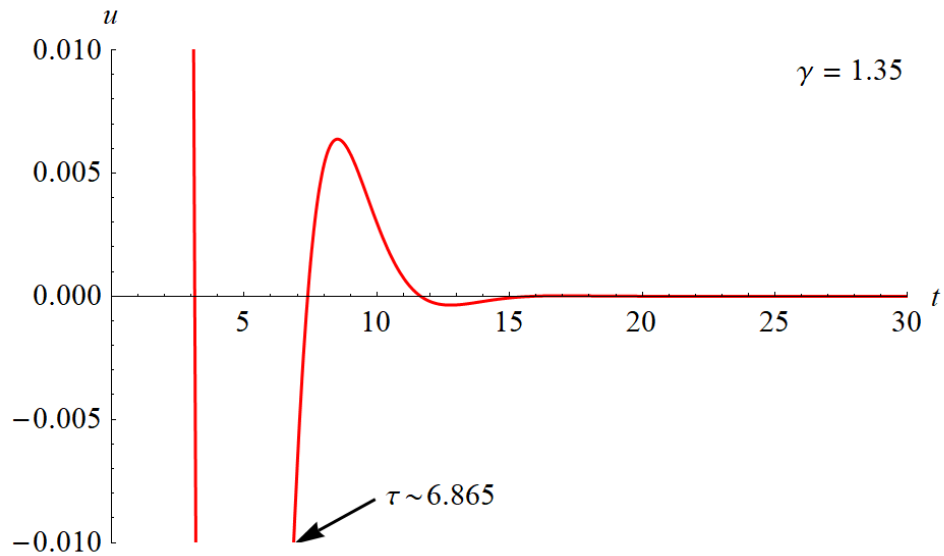
−0.010
5
10
15
20
25
30
t
τ ~ 6.731
γ = 1.30


u
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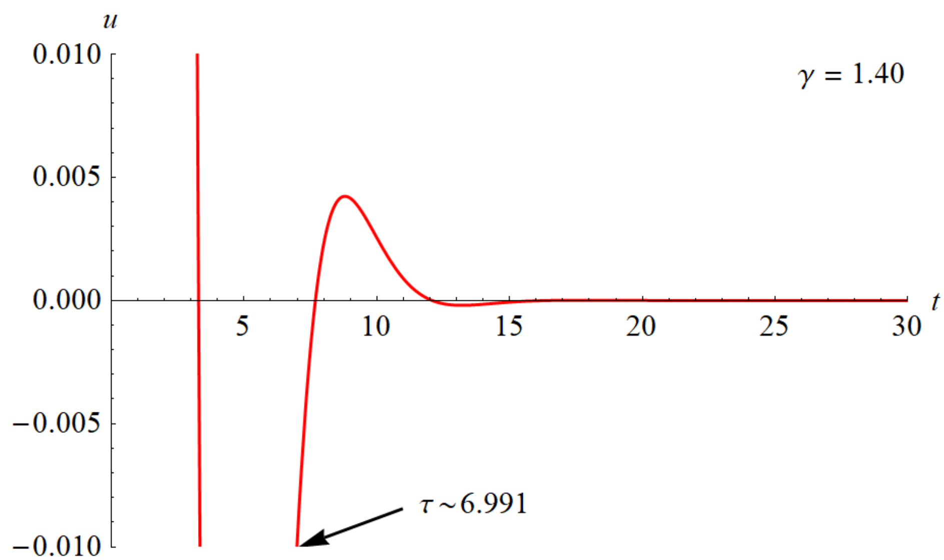
5
10
15
20
25
30
t
γ = 1.35
τ ~ 6.865


u
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25
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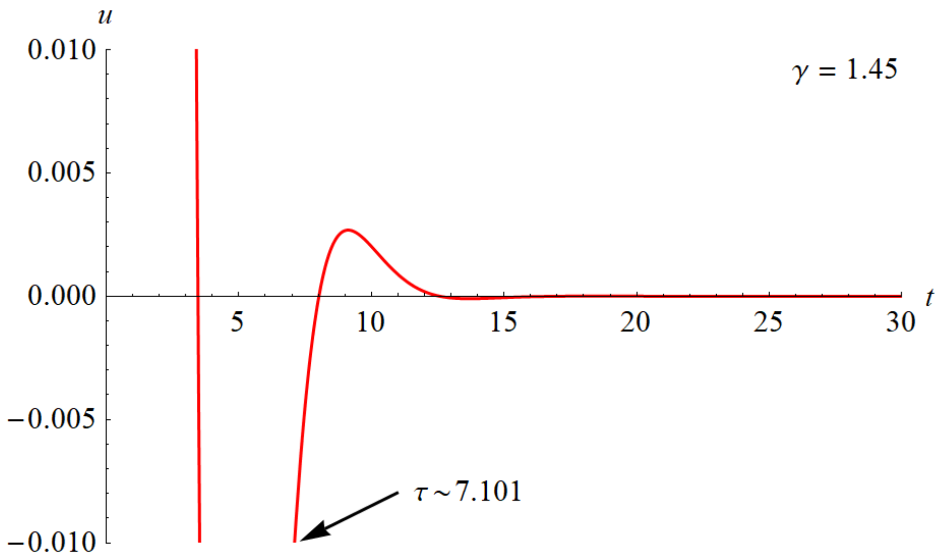
t
γ = 1.40
τ ~ 6.991


u
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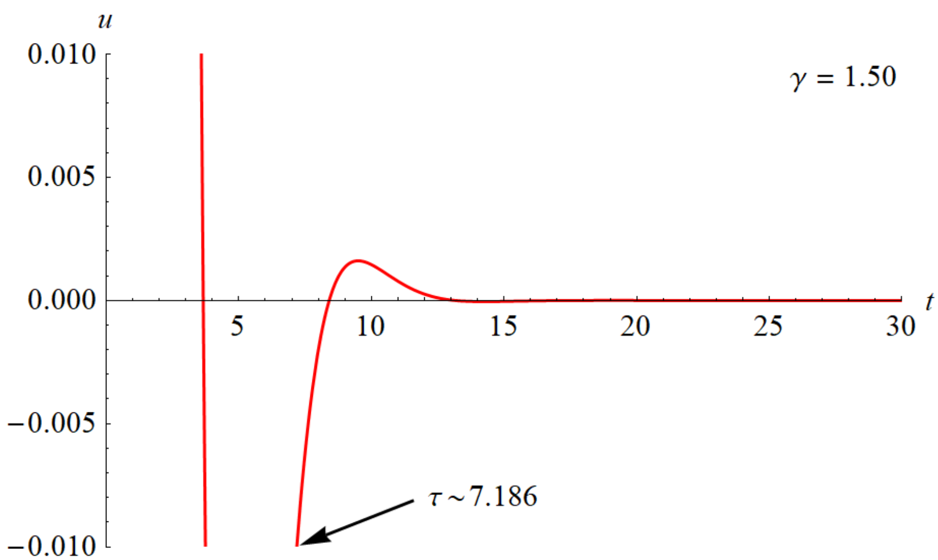
5
10
15
20
25
30
t
γ = 1.45
τ ~ 7.101


u
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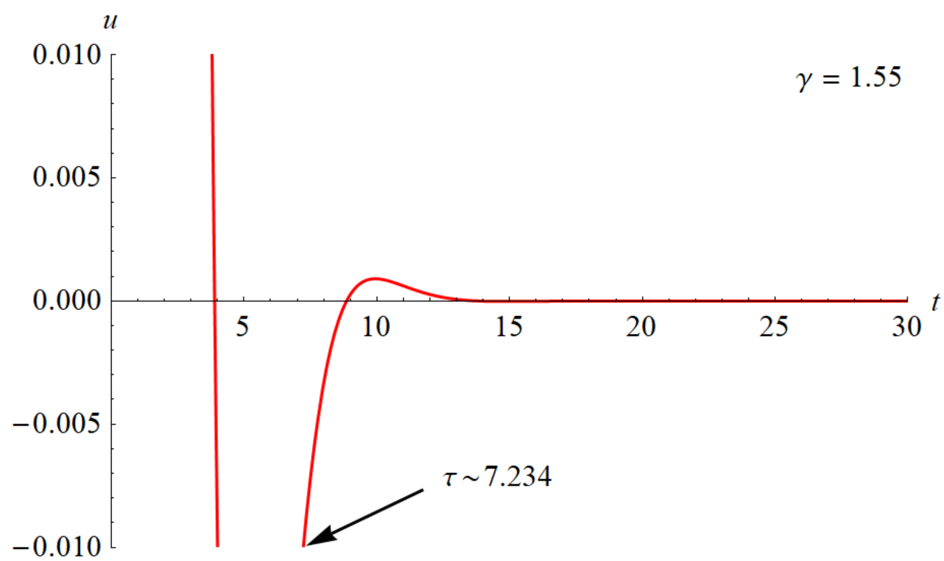
t
γ = 1.50
τ ~ 7.186


u
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5
10
15
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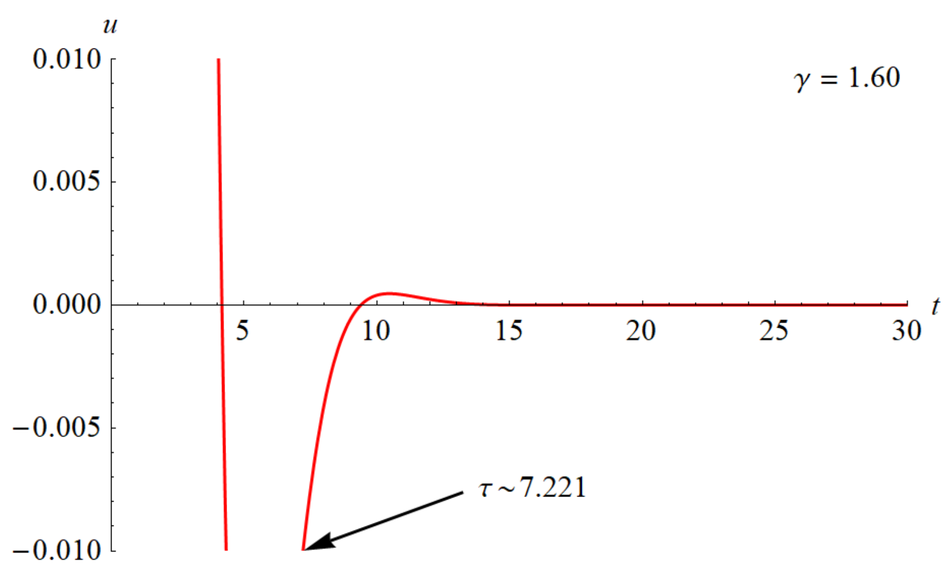
25
30
t
γ = 1.55
τ ~ 7.234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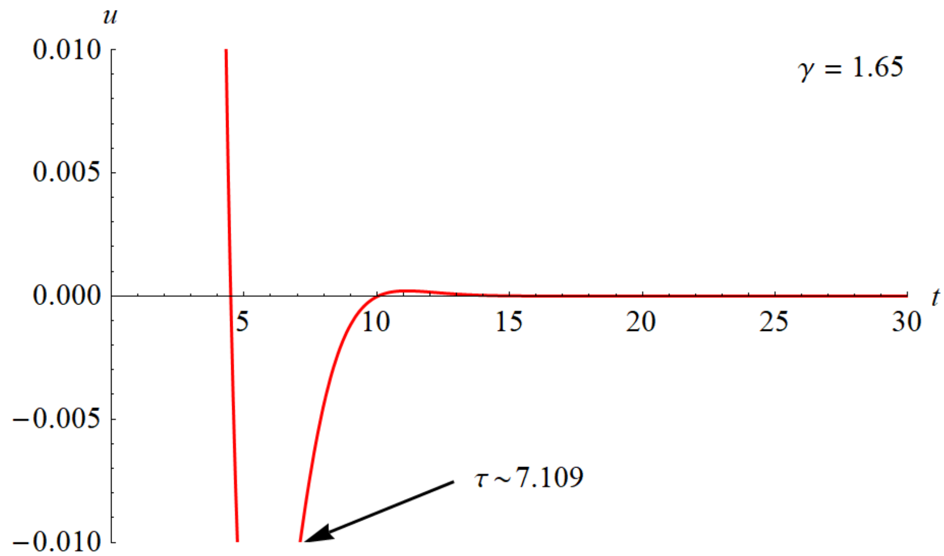
t
γ = 1.60
τ ~ 7.221


u
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5
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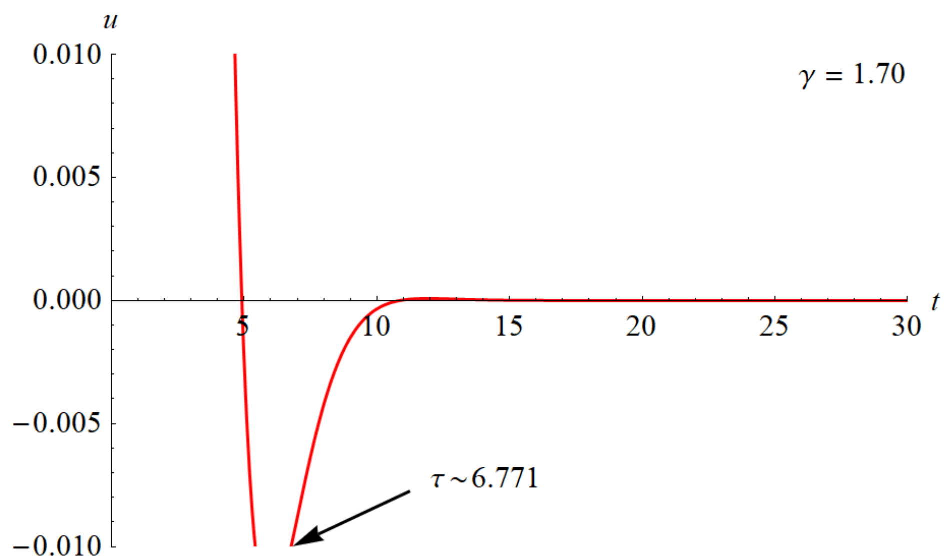
15
20
25
30
t
γ = 1.65
τ ~ 7.109


u
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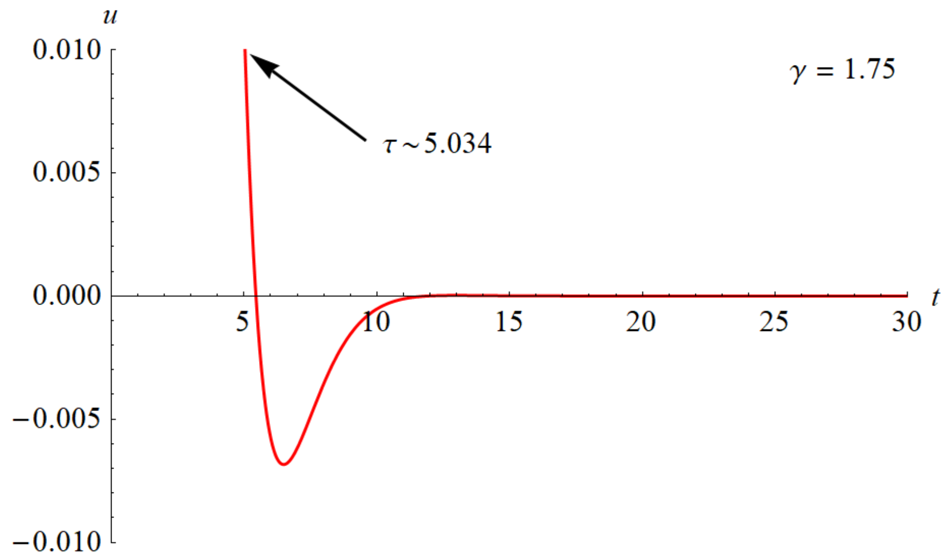
15
20
25
30
t
γ = 1.70
τ ~ 6.771



u
0.010
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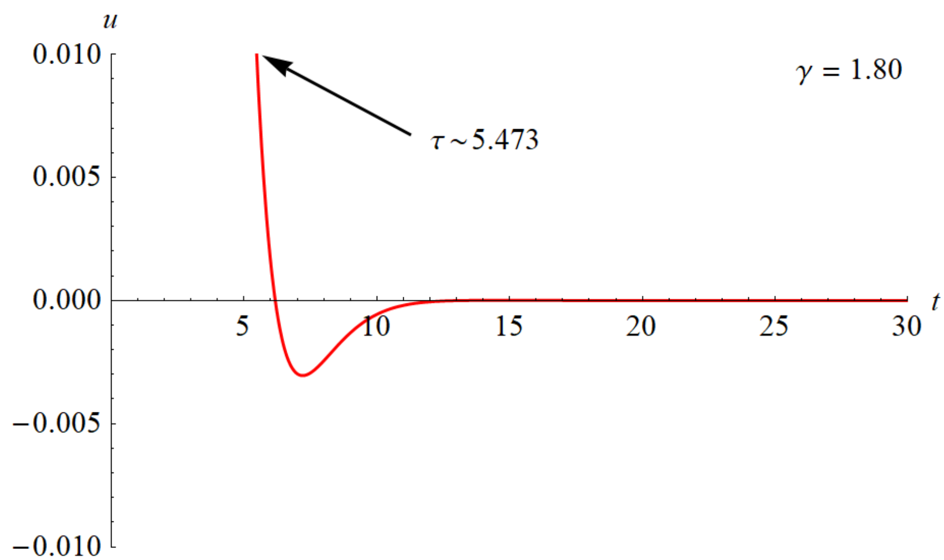
−0.010
5
10
15
20
25
30
t
γ = 1.75
τ ~ 5.034



u
0.010
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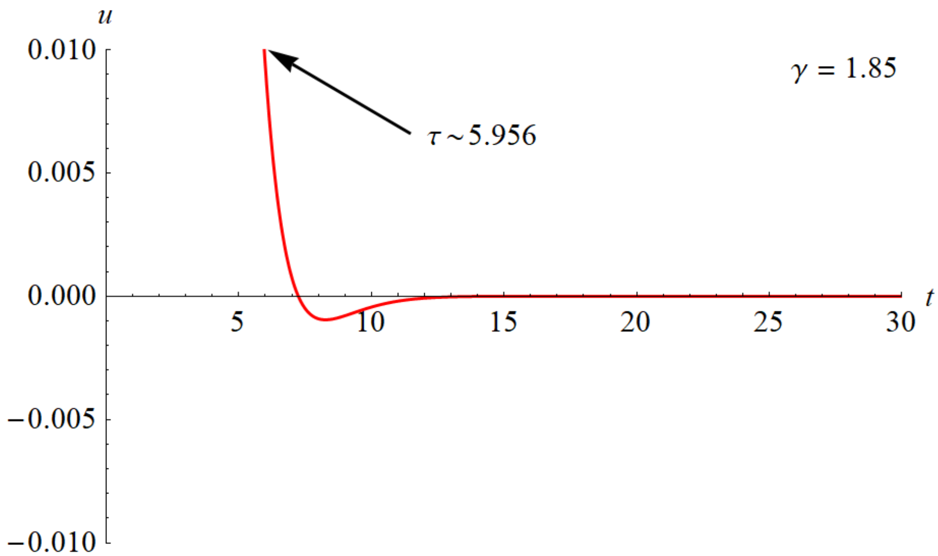
5
10
15
20
25
30
t
γ = 1.80
τ ~ 5.473


u
0.010
0.005
0.000
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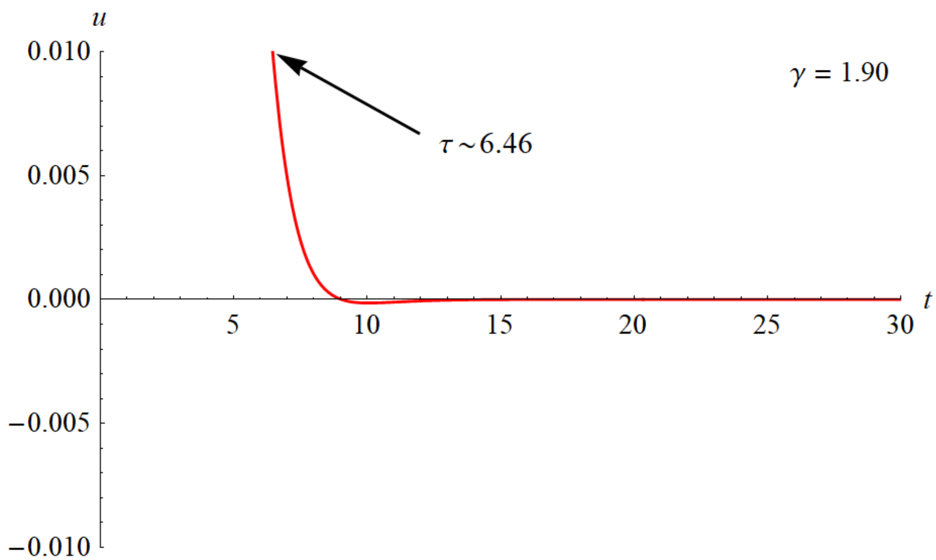
5
10
15
20
25
30
t
γ = 1.85
τ ~ 5.956


u
0.010
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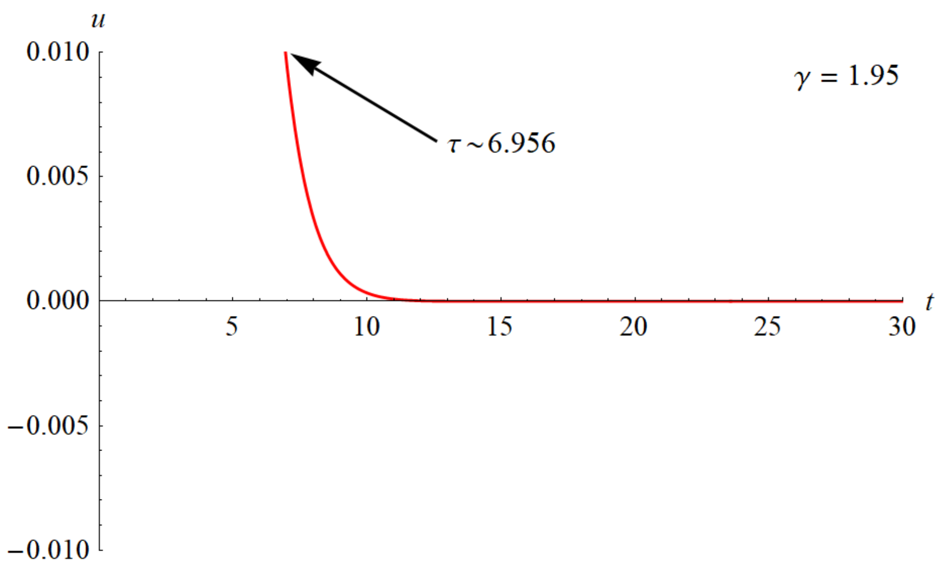
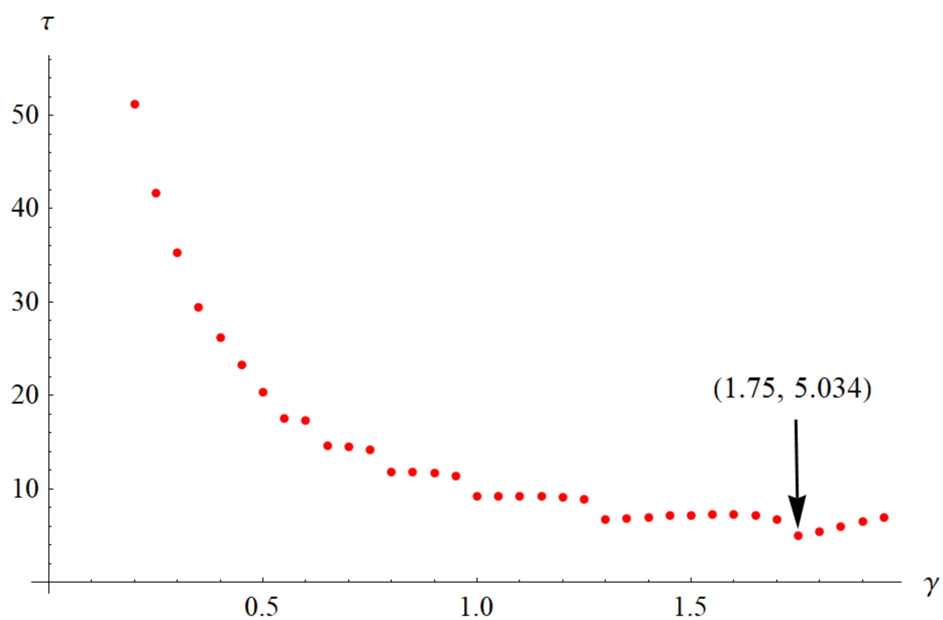
5
10
15
20
25
30
t
γ = 1.90
τ ~ 6.46



Now the points, $(0.1, 104.3)$, $(0.15, 69.76)$, $(0.2, 51.21)$, $(0.25, 41.72)$, $(0.3, 35.3)$, $(0.35, 29.46)$, $(0.4, 26.24)$, $(0.45, 23.25)$, $(0.5, 20.4)$, $(0.55, 17.59)$, $(0.6, 17.34)$, $(0.65, 14.67)$, $(0.7, 14.55)$, $(0.75, 14.24)$, $(0.8, 11.85)$, $(0.85, 11.81)$, $(0.9, 11.69)$, $(0.95, 11.41)$, $(1, 9.168)$, $(1.05, 9.212)$, $(1.1, 9.226)$, $(1.15, 9.195)$, $(1.2, 9.098)$, $(1.25, 8.872)$, $(1.3, 6.731)$, $(1.35, 6.865)$, $(1.4, 6.991)$, $(1.45, 7.101)$, $(1.5, 7.186)$, $(1.55, 7.234)$, $(1.6, 7.221)$, $(1.65, 7.109)$, $(1.7, 6.771)$, $(1.75, 5.034)$, $(1.8, 5.473)$, $(1.85, 5.956)$, $(1.9, 6.46)$, and $(1.95, 6.956)$, will be plotted. $(0.05, 210.8)$ is excluded because it's too high up compared to the others.



The lowest point is $\tau_{\text{min}} \approx 5.034$, and the value of γ that this occurs at is $\gamma_0 \approx 1.75$.

Return to the solution of the initial value problem.

$$u(t) = e^{-\gamma t/2}\left(2\cos\frac{\sqrt{4-\gamma^2}}{2}t + \frac{2\gamma}{\sqrt{4-\gamma^2}}\sin\frac{\sqrt{4-\gamma^2}}{2}t\right)$$

Introduce an amplitude R and a phase δ to combine the two sinusoidal terms into one.

$$\begin{aligned}
u(t) &= e^{-\gamma t/2}\left(R\cos\delta\cos\frac{\sqrt{4-\gamma^2}}{2}t + R\sin\delta\sin\frac{\sqrt{4-\gamma^2}}{2}t\right)\\
&= e^{-\gamma t/2}\left[R\cos\left(\frac{\sqrt{4-\gamma^2}}{2}t - \delta\right)\right]\\
&= Re^{-\gamma t/2}\cos\left(\frac{\sqrt{4-\gamma^2}}{2}t - \delta\right)
\end{aligned}$$

R and δ satisfy the following system of equations.

$$\begin{aligned}
R\cos\delta &= 2\\
R\sin\delta &= \frac{2\gamma}{\sqrt{4-\gamma^2}}
\end{aligned}$$

Square both sides of each equation

$$\begin{aligned}
R^2\cos^2\delta &= 4\\
R^2\sin^2\delta &= \frac{4\gamma^2}{4-\gamma^2}
\end{aligned}$$

and then add both sides to get R.

$$\begin{aligned}
R^2\cos^2\delta + R^2\sin^2\delta &= 4 + \frac{4\gamma^2}{4-\gamma^2}\\
R^2 &= \frac{16}{4-\gamma^2}\\
R &= \frac{4}{\sqrt{4-\gamma^2}}
\end{aligned}$$

The solution is then

$$u(t) = \frac{4}{\sqrt{4-\gamma^2}}e^{-\gamma t/2}\cos\left(\frac{\sqrt{4-\gamma^2}}{2}t - \delta\right).$$

Neglect the cosine term.

$$u(t) \approx \frac{4}{\sqrt{4-\gamma^2}}e^{-\gamma t/2}$$

To find when the amplitude reaches 0.01, set $u(t) = 0.01$ and then solve the resulting equation for $t = \tau$.

$$\frac{4}{\sqrt{4-\gamma^2}}e^{-\gamma\tau/2} = 0.01 \quad\rightarrow\quad e^{-\gamma\tau/2} = \frac{\sqrt{4-\gamma^2}}{400} \quad\rightarrow\quad -\frac{\gamma\tau}{2} = \ln\frac{\sqrt{4-\gamma^2}}{400}$$

Therefore,

$$\tau = \frac{2}{\gamma}\ln\frac{400}{\sqrt{4-\gamma^2}}.$$

Below is a side-by-side comparison of this approximation and the previous plot of points.