Problem 26

Consider the initial value problem

$$mu'' + \gamma u' + ku = 0, \qquad u(0) = u_0, \quad u'(0) = v_0.$$

Assume that $\gamma^2 < 4km$.

(a) Solve the initial value problem.

(b) Write the solution in the form $u(t) = R\exp(-\gamma t/2m)\cos(\mu t - \delta)$. Determine R in terms of m, γ, k, u_0, and v_0.

(c) Investigate the dependence of R on the damping coefficient γ for fixed values of the other parameters.

Solution

Since the coefficients in the ODE are constant and the ODE is homogeneous, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u' = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2 e^{rt}) + \gamma(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + \gamma r + k = 0$$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m} = \frac{-\gamma \pm i\sqrt{4mk - \gamma^2}}{2m} = -\frac{\gamma}{2m} \pm i\mu$$

$$r = \left\{-\frac{\gamma}{2m} - i\mu, -\frac{\gamma}{2m} + i\mu\right\}$$

Two solutions to the ODE are $u = e^{(-\gamma/2m - i\mu)t}$ and $u = e^{(-\gamma/2m + i\mu)t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
u(t) &= C_1 e^{(-\gamma/2m - i\mu)t} + C_2 e^{(-\gamma/2m + i\mu)t} \\
&= C_1 e^{-\gamma t/2m - i\mu t} + C_2 e^{-\gamma t/2m + i\mu t} \\
&= C_1 e^{-\gamma t/2m} e^{-i\mu t} + C_2 e^{-\gamma t/2m} e^{i\mu t} \\
&= C_1 e^{-\gamma t/2m}[\cos(-\mu t) + i\sin(-\mu t)] + C_2 e^{-\gamma t/2m}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-\gamma t/2m}[\cos(\mu t) - i\sin(\mu t)] + C_2 e^{-\gamma t/2m}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-\gamma t/2m}\cos\mu t - iC_1 e^{-\gamma t/2m}\sin\mu t + C_2 e^{-\gamma t/2m}\cos\mu t + iC_2 e^{-\gamma t/2m}\sin\mu t \\
&= (C_1 + C_2)e^{-\gamma t/2m}\cos\mu t + (-iC_1 + iC_2)e^{-\gamma t/2m}\sin\mu t \\
&= C_3 e^{-\gamma t/2m}\cos\mu t + C_4 e^{-\gamma t/2m}\sin\mu t \\
&= e^{-\gamma t/2m}(C_3\cos\mu t + C_4\sin\mu t)
\end{aligned}$$

Differentiate it with respect to t.

$$u'(t) = -\frac{\gamma}{2m}e^{-\gamma t/2m}(C_3\cos\mu t + C_4\sin\mu t) + e^{-\gamma t/2m}(-C_3\mu\sin\mu t + C_4\mu\cos\mu t)$$

Apply the initial conditions now to determine C_3 and C_4.

$$\begin{aligned} u(0) &= C_3 = u_0 \\ u'(0) &= -\frac{\gamma}{2m}C_3 + C_4\mu = v_0 \end{aligned}$$

Solving this system of equations yields

$$C_3 = u_0 \quad \text{and} \quad C_4 = \frac{1}{\mu}\left(v_0 + \frac{\gamma}{2m}u_0\right).$$

The solution to the initial value problem is then

$$\begin{aligned} u(t) &= e^{-\gamma t/2m}\left[u_0 \cos \mu t + \frac{1}{\mu}\left(v_0 + \frac{\gamma}{2m}u_0\right)\sin \mu t\right] \\ &= e^{-\gamma t/2m}\left[u_0 \cos \frac{\sqrt{4mk-\gamma^2}}{2m}t + \frac{2m}{\sqrt{4mk-\gamma^2}}\left(v_0 + \frac{\gamma}{2m}u_0\right)\sin \frac{\sqrt{4mk-\gamma^2}}{2m}t\right] \\ &= e^{-\gamma t/2m}\left[u_0 \cos \frac{\sqrt{4mk-\gamma^2}}{2m}t + \frac{1}{\sqrt{4mk-\gamma^2}}\left(2mv_0 + \gamma u_0\right)\sin \frac{\sqrt{4mk-\gamma^2}}{2m}t\right]. \end{aligned}$$

Now introduce an amplitude R and a phase δ that satisfy

$$\begin{aligned} R\cos\delta &= u_0 \\ R\sin\delta &= \frac{1}{\sqrt{4mk-\gamma^2}}\left(2mv_0 + \gamma u_0\right) \end{aligned}$$

so that the solution becomes

$$\begin{aligned} u(t) &= e^{-\gamma t/2m}\left[R\cos\delta\cos\frac{\sqrt{4mk-\gamma^2}}{2m}t + R\sin\delta\sin\frac{\sqrt{4mk-\gamma^2}}{2m}t\right] \\ &= Re^{-\gamma t/2m}\cos\left(\frac{\sqrt{4mk-\gamma^2}}{2m}t - \delta\right). \end{aligned}$$

Solving the system of equations for R and δ yields

$$R = \sqrt{u_0^2 + \frac{(2mv_0 + \gamma u_0)^2}{4km - \gamma^2}} \quad \text{and} \quad \tan\delta = \frac{1}{u_0\sqrt{4mk-\gamma^2}}\left(2mv_0 + \gamma u_0\right).$$

Below is a plot of R versus γ for $u_0 = 1$, $m = 1$, $v_0 = 1$, and $k = 1$.