Problem 28

The position of a certain undamped spring-mass system satisfies the initial value problem

$$u'' + 2u = 0, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Find the solution of this initial value problem.

(b) Plot u versus t and u' versus t on the same axes.

(c) Plot u' versus u; that is, plot $u(t)$ and $u'(t)$ parametrically with t as the parameter. This plot is known as a phase plot, and the uu'-plane is called the phase plane. Observe that a closed curve in the phase plane corresponds to a periodic solution $u(t)$. What is the direction of motion on the phase plot as t increases?

Solution

Since the coefficients are constant and this ODE is homogeneous, the solutions are of the form $u = e^{rt}$.

$$u = e^{rt} \quad \rightarrow \quad u = re^{rt} \quad \rightarrow \quad u'' = r^2 e^{rt}$$

Substitute these expressions into the ODE.

$$r^2 e^{rt} + 2(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + 2 = 0$$
$$r = \{-i\sqrt{2}, i\sqrt{2}\}$$

Two solutions to the ODE are $u = e^{-i\sqrt{2}t}$ and $u = e^{i\sqrt{2}t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned}
u(t) &= C_1 e^{-i\sqrt{2}t} + C_2 e^{i\sqrt{2}t} \\
&= C_1[\cos(-\sqrt{2}t) + i\sin(-\sqrt{2}t)] + C_2[\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)] \\
&= C_1[\cos(\sqrt{2}t) - i\sin(\sqrt{2}t)] + C_2[\cos(\sqrt{2}t) + i\sin(\sqrt{2}t)] \\
&= C_1 \cos\sqrt{2}t - iC_1 \sin\sqrt{2}t + C_2 \cos\sqrt{2}t + iC_2 \sin\sqrt{2}t \\
&= (C_1 + C_2)\cos\sqrt{2}t + (-iC_1 + iC_2)\sin\sqrt{2}t \\
&= C_3 \cos\sqrt{2}t + C_4 \sin\sqrt{2}t
\end{aligned}$$

Differentiate it with respect to t.

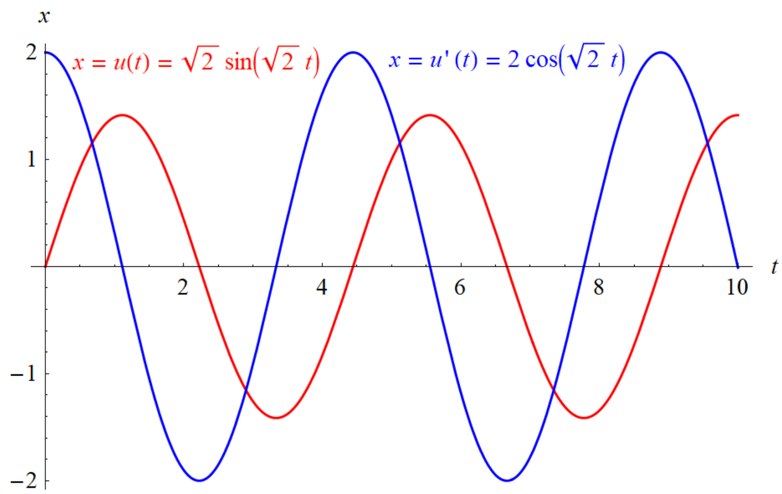
$$u'(t) = -C_3\sqrt{2}\sin\sqrt{2}t + C_4\sqrt{2}\cos\sqrt{2}t$$

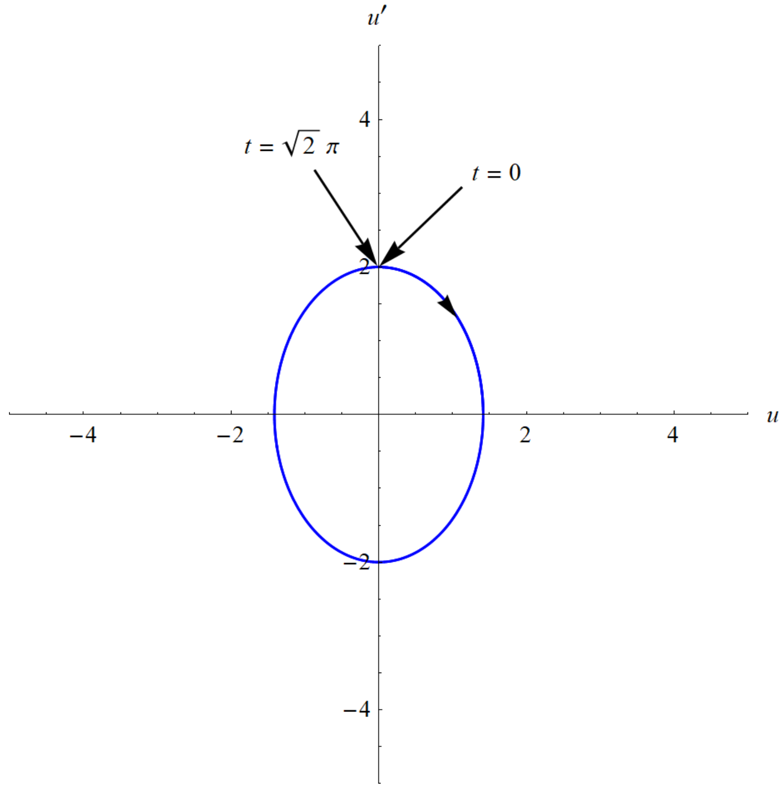
Apply the initial conditions here to determine C_3 and C_4.

$$u(0) = C_3 = 0$$
$$u'(0) = C_4\sqrt{2} = 2$$

Solving this system of equations yields $C_3 = 0$ and $C_4 = \sqrt{2}$. Therefore, the solution to the initial value problem is

$$u(t) = \sqrt{2}\sin\sqrt{2}t.$$


x
t
2
1
−1
−2
2
4
6
8
10
x = u(t) = √2 sin(√2 t)
x = u' (t) = 2 cos(√2 t)


u′
u
4
2
−2
−4
−4
−2
2
4
t = √2 π
t = 0