Problem 32

In the spring-mass system of Problem 31, suppose that the spring force is not given by Hooke's law but instead satisfies the relation

$$F_s = -(ku + \epsilon u^3),$$

where $k > 0$ and ϵ is small but may be of either sign. The spring is called a hardening spring if $\epsilon > 0$ and a softening spring if $\epsilon < 0$. Why are these terms appropriate?

(a) Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies the differential equation

$$mu'' + \gamma u' + ku + \epsilon u^3 = 0.$$

Suppose that the initial conditions are

$$u(0) = 0, \qquad u'(0) = 1.$$

In the remainder of this problem, assume that $m = 1$, $k = 1$, and $\gamma = 0$.

(b) Find $u(t)$ when $\epsilon = 0$ and also determine the amplitude and period of the motion.

(c) Let $\epsilon = 0.1$. Plot a numerical approximation to the solution. Does the motion appear to be periodic? Estimate the amplitude and period.

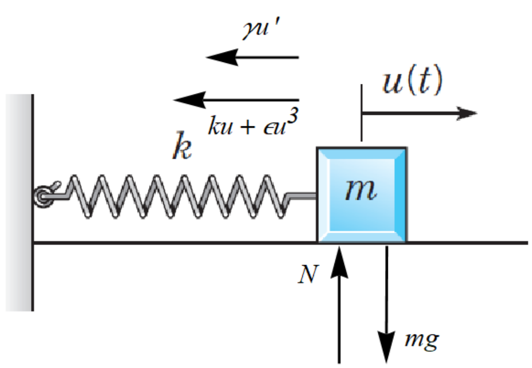
(d) Repeat part (c) for $\epsilon = 0.2$ and $\epsilon = 0.3$.

(e) Plot your estimated values of the amplitude A and the period T versus ϵ. Describe the way in which A and T, respectively, depend on ϵ.

(f) Repeat parts (c), (d), and (e) for negative values of ϵ.

Solution

Start by drawing a free-body diagram of the block. As a result of moving it to the right a distance u, two forces $ku + \epsilon u^3$ and $\gamma u'$ act to the left.



Apply Newton's second law in the u-direction.

$$\sum F_u = ma_u$$

$$-\gamma u' - (ku + \epsilon u^3) = ma_u$$

Use the fact that the acceleration is the second derivative of position.

$$-\gamma u' - ku - \epsilon u^3 = mu''$$

Bring all terms to one side.

$$mu'' + \gamma u' + ku + \epsilon u^3 = 0$$

Assume now that $m = 1$, $k = 1$, and $\gamma = 0$.

$$u'' + u + \epsilon u^3 = 0 \tag{1}$$

If $\epsilon = 0$, then this equation reduces to

$$u'' + u = 0,$$

which has the general solution,

$$u(t) = C_1 \cos t + C_2 \sin t.$$

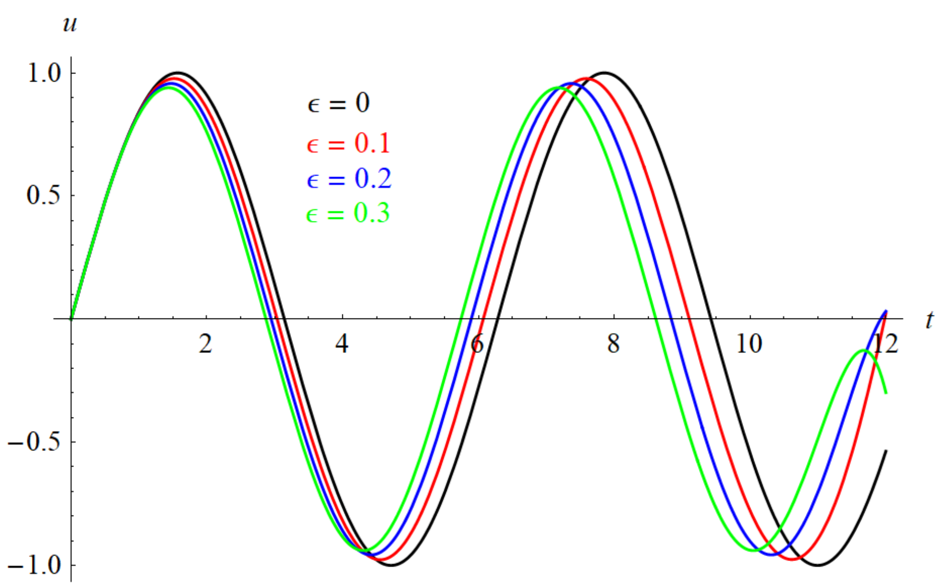
Apply the initial conditions, $u(0) = 0$ and $u'(0) = 1$, to determine C_1 and C_2.

$$u(0) = C_1 = 0$$
$$u'(0) = C_2 = 1$$

The solution for the case $\epsilon = 0$ is then

$$u(t) = \sin t.$$

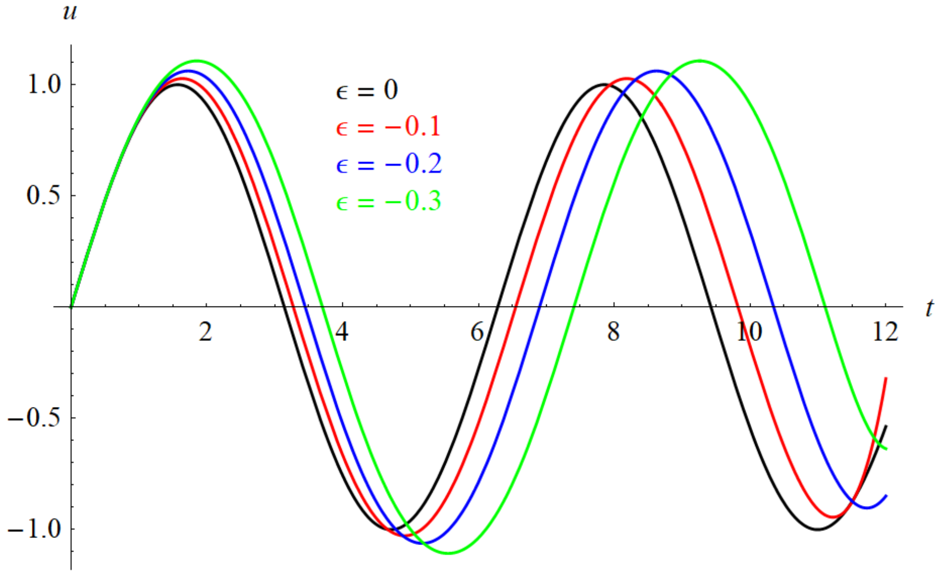
It has an amplitude $A = 1$ and a period $T = 2\pi$. Below it appears in black and is compared to numerical solutions of equation (1) for other values of ϵ.



Solutions for positive values of ϵ are periodic only for about 10 seconds before they diverge. By inspecting the graph, we note the following periods and amplitudes.

$$\epsilon = 0.1 \quad A \approx 0.977 \quad T \approx 6.07$$
$$\epsilon = 0.2 \quad A \approx 0.9569 \quad T \approx 5.894$$
$$\epsilon = 0.3 \quad A \approx 0.9395 \quad T \approx 5.742$$

Now we consider negative values of ϵ.



Solutions for positive values of ϵ are periodic only for about 12 seconds before they diverge. By inspecting the graph, we note the following periods and amplitudes.

$$\begin{aligned} &\epsilon = -0.1 \quad A \approx 1.027 \quad T \approx 6.55 \\ &\epsilon = -0.2 \quad A \approx 1.061 \quad T \approx 6.9 \\ &\epsilon = -0.3 \quad A \approx 1.107 \quad T \approx 7.406 \end{aligned}$$

Below are plots of A versus ϵ and T versus ϵ.