Problem 32

In the spring-mass system of Problem 31, suppose that the spring force is not given by Hooke's law but instead satisfies the relation

$$F_s = -(ku + \epsilon u^3),$$

where $k > 0$ and ϵ is small but may be of either sign. The spring is called a hardening spring if $\epsilon > 0$ and a softening spring if $\epsilon < 0$. Why are these terms appropriate?

(a) Show that the displacement $u(t)$ of the mass from its equilibrium position satisfies the differential equation

$$mu'' + \gamma u' + ku + \epsilon u^3 = 0.$$

Suppose that the initial conditions are

$$u(0) = 0, \qquad u'(0) = 1.$$

In the remainder of this problem, assume that $m = 1$, $k = 1$, and $\gamma = 0$.

(b) Find $u(t)$ when $\epsilon = 0$ and also determine the amplitude and period of the motion.

(c) Let $\epsilon = 0.1$. Plot a numerical approximation to the solution. Does the motion appear to be periodic? Estimate the amplitude and period.

(d) Repeat part (c) for $\epsilon = 0.2$ and $\epsilon = 0.3$.

(e) Plot your estimated values of the amplitude A and the period T versus ϵ. Describe the way in which A and T, respectively, depend on ϵ.

(f) Repeat parts (c), (d), and (e) for negative values of ϵ.