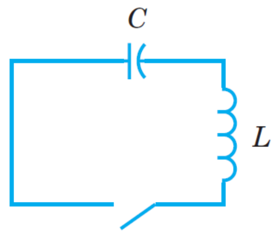
Problem 8

A series circuit has a capacitor of 0.25×10^{-6} F and an inductor of 1 H. If the initial charge on the capacitor is 10^{-6} C and there is no initial current, find the charge Q on the capacitor at any time t.

Solution

Start by drawing an LC series circuit.



Assume that the circuit is closed at $t = 0$. Apply Faraday's law to obtain the governing ODE for the current.

$$\sum V = -L\frac{di}{dt}$$

The only potential drop occurs over the capacitor.

$$\frac{q}{C} = -L\frac{di}{dt}$$

Write $i = dq/dt = q'$.

$$\frac{q}{C} = -Lq''$$

Bring Lq'' to the left side and divide both sides by L.

$$q'' + \frac{1}{LC}q = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $q = e^{rt}$.

$$q = e^{rt} \quad \rightarrow \quad q' = re^{rt} \quad \rightarrow \quad q'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2e^{rt} + \frac{1}{LC}(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^2 + \frac{1}{LC} = 0$$

$$r^2 = -\frac{1}{LC}$$

$$r = \pm i\frac{1}{\sqrt{LC}} = \pm i\omega$$

Two solutions to equation (1) are then $q = e^{-i\omega t}$ and $q = e^{i\omega t}$. By the principle of superposition, the general solution is a linear combination of these two.

$$\begin{aligned} q(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\ &= C_1[\cos(-\omega t) + i\sin(-\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1\cos\omega t - iC_1\sin\omega t + C_2\cos\omega t + iC_2\sin\omega t \\ &= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\ &= C_3\cos\omega t + C_4\sin\omega t \end{aligned}$$

Take a derivative of it with respect to t.

$$q'(t) = -\omega C_3\sin\omega t + \omega C_4\cos\omega t$$

Now apply the initial conditions, $q(0) = 10^{-6}$ and $q'(0) = 0$, to determine C_3 and C_4.

$$\begin{aligned} q(0) &= C_3 = 10^{-6} \\ q'(0) &= \omega C_4 = 0 \quad \rightarrow \quad C_4 = 0 \end{aligned}$$

Therefore,

$$\begin{aligned} q(t) &= 10^{-6}\cos\omega t \\ &= 10^{-6}\cos\frac{t}{\sqrt{LC}} \\ &= 10^{-6}\cos 2000t. \end{aligned}$$

Note that the charge $q(t)$ is in Coulombs (C).