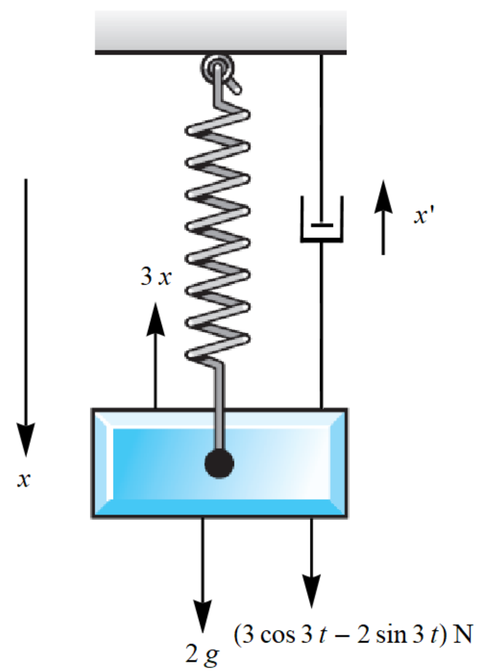
Problem 12

A spring-mass system has a spring constant of 3 N/m. A mass of 2 kg is attached to the spring, and the motion takes place in a viscous fluid that offers a resistance numerically equal to the magnitude of the instantaneous velocity. If the system is driven by an external force of $(3\cos 3t - 2\sin 3t)$ N, determine the steady state response. Express your answer in the form $R\cos(\omega t - \delta)$.

Solution

Start by drawing a free-body diagram of the mass.



Apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-x' - 3x + 2g + 3\cos 3t - 2\sin 3t = 2a_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-x' - 3x + 2g + 3\cos 3t - 2\sin 3t = 2x''$$

$$2x'' + x' + 3x = 2g + 3\cos 3t - 2\sin 3t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$2x_c'' + x_c' + 3x_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$2(r^2 e^{rt}) + re^{rt} + 3(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$2r^2 + r + 3 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(2)(3)}}{2(2)} = \frac{-1 \pm \sqrt{-23}}{4} = \frac{-1 \pm i\sqrt{23}}{4} = -\frac{1}{4} \pm i\mu$$

Two solutions to equation (1) are then

$$x_c = e^{(-1/4 - i\mu)t} \quad \text{and} \quad x_c = e^{(-1/4 + i\mu)t}.$$

By the principle of superposition, the general solution for x_c is a linear combination of these two.

$$\begin{aligned}
x_c(t) &= C_1 e^{(-1/4 - i\mu)t} + C_2 e^{(-1/4 + i\mu)t} \\
&= C_1 e^{-t/4 - i\mu t} + C_2 e^{-t/4 + i\mu t} \\
&= C_1 e^{-t/4} e^{-i\mu t} + C_2 e^{-t/4} e^{i\mu t} \\
&= C_1 e^{-t/4}[\cos(-\mu t) + i\sin(-\mu t)] + C_2 e^{-t/4}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-t/4}[\cos(\mu t) - i\sin(\mu t)] + C_2 e^{-t/4}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-t/4}\cos\mu t - iC_1 e^{-t/4}\sin\mu t + C_2 e^{-t/4}\cos\mu t + iC_2 e^{-t/4}\sin\mu t \\
&= (C_1 + C_2)e^{-t/4}\cos\mu t + (-iC_1 + iC_2)e^{-t/4}\sin\mu t \\
&= C_3 e^{-t/4}\cos\mu t + C_4 e^{-t/4}\sin\mu t \\
&= e^{-t/4}(C_3\cos\mu t + C_4\sin\mu t)
\end{aligned}$$

On the other hand, the particular solution satisfies

$$2x_p'' + x_p' + 3x_p = 2g + 3\cos 3t - 2\sin 3t.$$

For the first term on the right side, we'll include a constant A in the trial solution. To account for the sine and cosine terms, we'll include $B\cos 3t + C\sin 3t$ in the trial solution. Substitute $x_p(t) = A + B\cos 3t + C\sin 3t$ into the equation to determine A and B and C.

$$2(A+B\cos 3t+C\sin 3t)'' + (A+B\cos 3t+C\sin 3t)' + 3(A+B\cos 3t+C\sin 3t) = 2g+3\cos 3t-2\sin 3t$$

$$2(-9B\cos 3t-9C\sin 3t) + (-3B\sin 3t+3C\cos 3t) + 3(A+B\cos 3t+C\sin 3t) = 2g+3\cos 3t-2\sin 3t$$

$$3A + (-18B + 3C + 3B)\cos 3t + (-18C - 3B + 3C)\sin 3t = 2g + 3\cos 3t - 2\sin 3t$$

Matching the coefficients, we have

$$\begin{aligned}
3A &= 2g \\
-18B + 3C + 3B &= 3 \\
-18C - 3B + 3C &= -2.
\end{aligned}$$

Solving this system yields

$$A = \frac{2g}{3} \quad \text{and} \quad B = -\frac{1}{6} \quad \text{and} \quad C = \frac{1}{6}.$$

The particular solution is then

$$x_p(t) = \frac{2g}{3} - \frac{1}{6}\cos 3t + \frac{1}{6}\sin 3t,$$

which means the general solution is

$$\begin{aligned} x(t) &= e^{-t/4}(C_3 \cos \mu t + C_4 \sin \mu t) + \frac{2g}{3} - \frac{1}{6}\cos 3t + \frac{1}{6}\sin 3t \\ &= e^{-t/4}\left(C_3 \cos \frac{\sqrt{23}}{4}t + C_4 \sin \frac{\sqrt{23}}{4}t\right) + \underbrace{\frac{2g}{3} - \frac{1}{6}\cos 3t + \frac{1}{6}\sin 3t}_{\text{steady state}}. \end{aligned}$$

To combine the two sinusoidal terms into one, introduce an amplitude R and a phase δ.

$$\begin{aligned} x(t) &= e^{-t/4}\left(C_3 \cos \frac{\sqrt{23}}{4}t + C_4 \sin \frac{\sqrt{23}}{4}t\right) + \frac{2g}{3} + R\cos\delta\cos 3t + R\sin\delta\sin 3t \\ &= e^{-t/4}\left(C_3 \cos \frac{\sqrt{23}}{4}t + C_4 \sin \frac{\sqrt{23}}{4}t\right) + \frac{2g}{3} + R\cos(3t - \delta) \end{aligned}$$

R and δ satisfy the following system of equations.

$$R\cos\delta = -\frac{1}{6} \tag{2}$$

$$R\sin\delta = \frac{1}{6} \tag{3}$$

To find R, square both sides of each equation and then add the respective sides together.

$$\begin{aligned} R &= \sqrt{\left(-\frac{1}{6}\right)^2 + \left(\frac{1}{6}\right)^2} \\ &= \frac{1}{3\sqrt{2}} \end{aligned}$$

To find δ, divide the respective sides of equation (3) by those of equation (2).

$$\tan\delta = -1$$

$$\begin{aligned} \delta &= \tan^{-1}(-1) \\ &= -\tan^{-1} 1 + \pi \\ &= -\frac{\pi}{4} + \pi \\ &= \frac{3\pi}{4} \end{aligned}$$

Therefore,

$$x(t) = e^{-t/4}\left(C_3 \cos \frac{\sqrt{23}}{4}t + C_4 \sin \frac{\sqrt{23}}{4}t\right) + \frac{2g}{3} + \frac{1}{3\sqrt{2}}\cos\left(3t - \frac{3\pi}{4}\right).$$