Problem 13

In this problem we ask you to supply some of the details in the analysis of a forced damped oscillator.

(a) Derive Eqs. (10), (11), and (12) for the steady state solution of Eq. (8).

(b) Derive the expression in Eq. (13) for Rk/F_0.

(c) Show that ω^2_{max} and R_{max} are given by Eqs. (14) and (15), respectively.

Solution

Eq. (8) in the text is the equation of motion for a mass m attached to a spring with constant k and a dashpot with damping coefficient γ that is subject to an external force $F_0 \cos \omega t$.

$$mu'' + \gamma u' + ku = F_0 \cos \omega t \tag{8}$$

The steady-state solution is the particular solution for this ODE.

$$mu_p'' + \gamma u_p' + ku_p = F_0 \cos \omega t$$

Since there are both even and odd derivatives on the left side, we include sine and cosine terms in the trial solution. Plug in $u_p(t) = A\cos\omega t + B\sin\omega t$ to determine A and B.

$$m(A\cos\omega t + B\sin\omega t)'' + \gamma(A\cos\omega t + B\sin\omega t)' + k(A\cos\omega t + B\sin\omega t) = F_0\cos\omega t$$

Evaluate the derivatives.

$$m(-A\omega^2\cos\omega t - B\omega^2\sin\omega t) + \gamma(-A\omega\sin\omega t + B\omega\cos\omega t) + k(A\cos\omega t + B\sin\omega t) = F_0\cos\omega t$$

$$(-mA\omega^2 + \gamma B\omega + kA)\cos\omega t + (-mB\omega^2 - \gamma A\omega + kB)\sin\omega t = F_0\cos\omega t$$

Match the coefficients to get a system of two equations for A and B.

$$\begin{aligned} -mA\omega^2 + \gamma B\omega + kA &= F_0 \\ -mB\omega^2 - \gamma A\omega + kB &= 0 \end{aligned}$$

Solving it yields

$$A = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2} \quad \text{and} \quad B = \frac{F_0\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}.$$

Therefore, the steady-state response is

$$u_p(t) = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2}\cos\omega t + \frac{F_0\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}\sin\omega t.$$

Introduce an amplitude R and a phase δ to combine these two sinusoidal terms into one.

$$\begin{aligned} u_p(t) &= R\cos\delta\cos\omega t + R\sin\delta\sin\omega t \\ &= R\cos(\omega t - \delta) \end{aligned}$$

R and δ satisfy the following system of equations.

$$R\cos\delta = \frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2} \tag{1}$$

$$R\sin\delta = \frac{F_0\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2} \tag{2}$$

To find R, square both sides of each equation

$$R^2\cos^2\delta = \left[\frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2}\right]^2$$

$$R^2\sin^2\delta = \left[\frac{F_0\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}\right]^2$$

and then add the respective sides.

$$R^2(\cos^2\delta + \sin^2\delta) = \left[\frac{F_0(k - m\omega^2)}{(k - m\omega^2)^2 + \gamma^2\omega^2}\right]^2 + \left[\frac{F_0\gamma\omega}{(k - m\omega^2)^2 + \gamma^2\omega^2}\right]^2$$

Simplify both sides.

$$R^2 = \frac{F_0^2}{(k - m\omega^2)^2 + \gamma^2\omega^2}$$

$$\begin{aligned} R &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + \gamma^2\omega^2}} \\ &= \frac{F_0}{\sqrt{m^2\left(\frac{k}{m} - \omega^2\right)^2 + \gamma^2\omega^2}} \end{aligned}$$

If we set $\omega_0^2 = k/m$, then we get the first desired result.

$$\boxed{R = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}}$$

Using ω_0^2 for k/m, equations (1) and (2) become

$$R\cos\delta = \frac{F_0 m(\omega_0^2 - \omega^2)}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

$$R\sin\delta = \frac{F_0\gamma\omega}{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}$$

Divide both sides of each equation by R to get the second and third desired results.

$$\boxed{\cos\delta = \frac{m(\omega_0^2 - \omega^2)}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}}$$

$$\boxed{\sin\delta = \frac{\gamma\omega}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}}$$

Use the boxed formula for R to determine Rk/F_0.

$$\begin{aligned}
\frac{Rk}{F_0} &= \frac{k}{F_0}\frac{F_0}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \\
&= \frac{k}{\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \\
&= \frac{1}{\frac{1}{k}\sqrt{m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} \\
&= \frac{1}{\sqrt{\frac{m^2}{k^2}(\omega_0^2-\omega^2)^2+\frac{\gamma^2\omega^2}{k^2}}} \\
&= \frac{1}{\sqrt{\frac{m^2}{k^2}\omega_0^4\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{\gamma^2}{mk}\frac{m\omega^2}{k}}} \\
&= \frac{1}{\sqrt{\frac{m^2}{k^2}\frac{k^2}{m^2}\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\frac{\gamma^2}{mk}\frac{\omega^2}{\frac{k}{m}}}}
\end{aligned}$$

Therefore, the fourth desired result is

$$\boxed{\frac{Rk}{F_0} = \frac{1}{\sqrt{\left(1-\frac{\omega^2}{\omega_0^2}\right)^2+\Gamma\frac{\omega^2}{\omega_0^2}}},}$$

where $\Gamma = \gamma^2/(mk)$. To find the value of ω that maximizes R, differentiate the boxed formula for R with respect to ω

$$R'(\omega) = -\frac{F_0}{2[m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{3/2}}[2m^2(\omega_0^2-\omega^2)(-2\omega)+2\gamma^2\omega]$$

and then set it equal to zero.

$$-\frac{F_0}{2[m^2(\omega_0^2-\omega^2)^2+\gamma^2\omega^2]^{3/2}}[2m^2(\omega_0^2-\omega^2)(-2\omega)+2\gamma^2\omega] = 0$$

Solve for ω.

$$\begin{gathered}
2m^2(\omega_0^2-\omega^2)(-2\omega)+2\gamma^2\omega = 0 \\
2\omega[-2m^2(\omega_0^2-\omega^2)+\gamma^2] = 0
\end{gathered}$$

Use the zero product theorem.

$$\begin{aligned}
2\omega = 0 \quad &\text{or} \quad -2m^2(\omega_0^2-\omega^2)+\gamma^2 = 0 \\
\omega = 0 \quad &\text{or} \quad \omega_0^2-\omega^2 = \frac{\gamma^2}{2m^2} \\
&\qquad \omega^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}
\end{aligned}$$

Note that ω^2 can also be written as

$$\begin{aligned}\omega^2 &= \omega_0^2\left(1-\frac{\gamma^2}{2m^2\omega_0^2}\right)\\ &= \omega_0^2\left(1-\frac{\gamma^2}{2m^2\frac{k}{m}}\right)\\ &= \omega_0^2\left(1-\frac{\gamma^2}{2km}\right)\\ &= \omega_0^2\left(1-\frac{\Gamma}{2}\right).\end{aligned}$$

Therefore, the fifth desired result is

$$\boxed{\omega_{\text{max}}^2 = \omega_0^2 - \frac{\gamma^2}{2m^2}.}$$

Plugging this formula for ω^2 into the boxed formula for R yields the maximum value for R.

$$\begin{aligned}R_{\text{max}} &= \frac{F_0}{\sqrt{m^2(\omega_0^2-\omega_{\text{max}}^2)^2+\gamma^2\omega_{\text{max}}^2}}\\ &= \frac{F_0}{\sqrt{m^2\left(\frac{\gamma^2}{2m^2}\right)^2+\gamma^2\left(\omega_0^2-\frac{\gamma^2}{2m^2}\right)}}\\ &= \frac{F_0}{\sqrt{\frac{\gamma^4}{4m^2}+\gamma^2\omega_0^2-\frac{\gamma^4}{2m^2}}}\\ &= \frac{F_0}{\sqrt{\gamma^2\omega_0^2-\frac{\gamma^4}{4m^2}}}\\ &= \frac{F_0}{\gamma\omega_0\sqrt{1-\frac{\gamma^2}{4m^2\omega_0^2}}}\\ &= \frac{F_0}{\gamma\omega_0\sqrt{1-\frac{\gamma^2}{4m^2\frac{k}{m}}}}\end{aligned}$$

Therefore, the sixth and final desired result is

$$\boxed{R_{\text{max}} = \frac{F_0}{\gamma\omega_0\sqrt{1-\frac{\gamma^2}{4mk}}}.}$$