Problem 15

Find the solution of the initial value problem

$$u'' + u = F(t), \qquad u(0) = 0, \quad u'(0) = 0,$$

where

$$F(t) = \begin{cases} F_0 t, & 0 \le t \le \pi, \\ F_0(2\pi - t), & \pi < t \le 2\pi, \\ 0, & 2\pi < t. \end{cases}$$

Hint: Treat each time interval separately, and match the solutions in the different intervals by requiring u and u' to be continuous functions of t.

Solution

Split up the ODE over the three time intervals that the forcing function is defined on.

$$\begin{aligned} u'' + u &= F_0 t, & 0 \le t \le \pi \\ u'' + u &= F_0(2\pi - t), & \pi < t \le 2\pi \\ u'' + u &= 0, & t > 2\pi \end{aligned}$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$u(t) = u_c(t) + u_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_c'' + u_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u_c = e^{rt}$.

$$u_c = e^{rt} \quad \rightarrow \quad u_c' = re^{rt} \quad \rightarrow \quad u_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2 e^{rt} + e^{rt} = 0$$

Divide both sides by e^{rt}.

$$r^2 + 1 = 0$$

$$r = \{-i, i\}$$

Two solutions to equation (1) are $u_c = e^{-it}$ and $u_c = e^{it}$. By the principle of superposition, the general solution for u_c is a linear combination of these two.

$$\begin{aligned} u_c(t) &= C_1 e^{-it} + C_2 e^{it} \\ &= C_1[\cos(-t) + i\sin(-t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1[\cos(t) - i\sin(t)] + C_2[\cos(t) + i\sin(t)] \\ &= C_1 \cos t - iC_1 \sin t + C_2 \cos t + iC_2 \sin t \\ &= (C_1 + C_2)\cos t + (-iC_1 + iC_2)\sin t \\ &= C_3 \cos t + C_4 \sin t \end{aligned}$$

On the other hand, the particular solution satisfies

$$\begin{aligned} u_p'' + u_p &= F_0 t, && 0 \le t \le \pi \\ u_p'' + u_p &= 2\pi F_0 - F_0 t, && \pi < t \le 2\pi \\ u_p'' + u_p &= 0, && t > 2\pi. \end{aligned}$$

Plug in the trial solution $u_p(t) = A_1 + A_2 t$ to the first ODE and plug in the trial solution $u_p(t) = B_1 + B_2 t$ to the second ODE.

$$\begin{aligned} (A_1 + A_2 t)'' + (A_1 + A_2 t) &= F_0 t, && 0 \le t \le \pi \\ (B_1 + B_2 t)'' + (B_1 + B_2 t) &= 2\pi F_0 - F_0 t, && \pi < t \le 2\pi \end{aligned}$$

Evaluate the derivatives.

$$\begin{aligned} A_1 + A_2 t &= F_0 t, && 0 \le t \le \pi \\ B_1 + B_2 t &= 2\pi F_0 - F_0 t, && \pi < t \le 2\pi \end{aligned}$$

Match the coefficients to get A_1, A_2, B_1, and B_2.

$$\begin{aligned} A_1 &= 0 \\ A_2 &= F_0 \\ B_1 &= 2\pi F_0 \\ B_2 &= -F_0 \end{aligned}$$

Consequently, the general solution is

$$\begin{aligned} u(t) &= D_1 \cos t + D_2 \sin t + F_0 t, && 0 \le t \le \pi \\ u(t) &= D_3 \cos t + D_4 \sin t + 2\pi F_0 - F_0 t, && \pi < t \le 2\pi \\ u(t) &= D_5 \cos t + D_6 \sin t, && t > 2\pi. \end{aligned}$$

The two initial conditions, $u(0) = 0$ and $u'(0) = 0$, apply to the solution valid for $0 \le t \le \pi$. Use them to determine D_1 and D_2.

$$\begin{aligned} u(0) &= D_1 = 0 \\ u'(0) &= D_2 + F_0 = 0 \quad \rightarrow \quad D_2 = -F_0 \end{aligned}$$

The general solution becomes

$$\begin{aligned} u(t) &= -F_0 \sin t + F_0 t, && 0 \le t \le \pi \\ u(t) &= D_3 \cos t + D_4 \sin t + 2\pi F_0 - F_0 t, && \pi < t \le 2\pi \\ u(t) &= D_5 \cos t + D_6 \sin t, && t > 2\pi \end{aligned}$$

as a result. To determine D_3 and D_4, we require that the solution and its slope at $t = \pi$ are continuous.

$$\begin{aligned} \lim_{t \to \pi^-} u(t) &= \lim_{t \to \pi^+} u(t) \\ \lim_{t \to \pi^-} \frac{du}{dt} &= \lim_{t \to \pi^+} \frac{du}{dt} \end{aligned}$$

$$-F_0 \sin \pi + F_0 \pi = D_3 \cos \pi + D_4 \sin \pi + 2\pi F_0 - F_0 \pi$$
$$-F_0 \cos \pi + F_0 = -D_3 \sin \pi + D_4 \cos \pi - F_0$$

Solving this system of equations yields $D_3 = 0$ and $D_4 = -3F_0$. The general solution becomes

$$u(t) = -F_0 \sin t + F_0 t, \qquad 0 \le t \le \pi$$
$$u(t) = -3F_0 \sin t + 2\pi F_0 - F_0 t, \qquad \pi < t \le 2\pi$$
$$u(t) = D_5 \cos t + D_6 \sin t, \qquad t > 2\pi$$

as a result. To determine D_5 and D_6, we require that the solution and its slope at $t = 2\pi$ are continuous.

$$\lim_{t \to 2\pi^-} u(t) = \lim_{t \to 2\pi^+} u(t)$$
$$\lim_{t \to 2\pi^-} \frac{du}{dt} = \lim_{t \to 2\pi^+} \frac{du}{dt}$$

$$-3F_0 \sin 2\pi + 2\pi F_0 - 2F_0 \pi = D_5 \cos 2\pi + D_6 \sin 2\pi$$
$$-3F_0 \cos 2\pi - F_0 = -D_5 \sin 2\pi + D_6 \cos 2\pi$$

Solving this system of equations yields $D_5 = 0$ and $D_6 = -4F_0$. Therefore,

$$u(t) = \begin{cases} F_0(t - \sin t) & 0 \le t \le \pi \\ F_0(2\pi - t - 3\sin t) & \pi < t \le 2\pi \\ -4F_0 \sin t & t > 2\pi \end{cases}.$$

Below is a plot of $u(t)$ versus t for the special case that $F_0 = 1$.