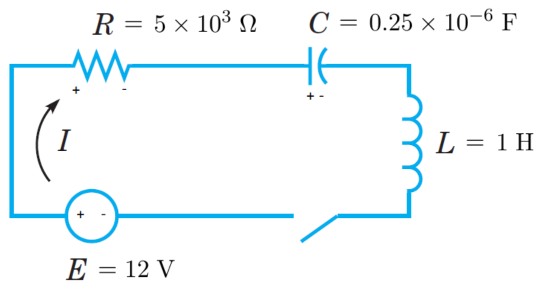
Problem 16

A series circuit has a capacitor of 0.25×10^{-6} F, a resistor of 5×10^3 Ω, and an inductor of 1 H. The initial charge on the capacitor is zero. If a 12-volt battery is connected to the circuit and the circuit is closed at $t = 0$, determine the charge on the capacitor at $t = 0.001$ s, at $t = 0.01$ s, and at any time t. Also determine the limiting charge as $t \to \infty$.

Solution

Start by drawing a schematic for the RLC series circuit.



Assume that the circuit is closed at $t = 0$. Apply Faraday's law to obtain the governing equation for the resulting current in the circuit.

$$\sum V = -L\frac{di}{dt}$$

There are three changes in potential that occur; the first occurs over the battery, the second occurs over the resistor, and the third occurs over the capacitor.

$$-12 + Ri + \frac{q}{C} = -L\frac{di}{dt}$$

Use the fact that $i = dq/dt = q'$.

$$-12 + Rq' + \frac{q}{C} = -Lq''$$

$$Lq'' + Rq' + \frac{q}{C} = 12$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$q(t) = q_c(t) + q_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$Lq_c'' + Rq_c' + \frac{q_c}{C} = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $q_c = e^{rt}$.

$$q_c = e^{rt} \quad \to \quad q_c' = re^{rt} \quad \to \quad q_c'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$L(r^2e^{rt}) + R(re^{rt}) + \frac{e^{rt}}{C} = 0$$

Divide both sides by e^{rt}/C.

$$LCr^2 + RCr + 1 = 0$$

$$r = \frac{-RC \pm \sqrt{R^2C^2 - 4LC}}{2LC} = -\frac{R}{2L} \pm \frac{\sqrt{R^2C^2 - 4LC}}{2LC}$$

$$r = \left\{-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}, -\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right\}$$

Two solutions to equation (1) are

$$q_c = \exp\left[\left(-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] \quad \text{and} \quad q_c = \exp\left[\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right].$$

By the principle of superposition, the general solution for q_c is a linear combination of these two.

$$q_c(t) = C_1 \exp\left[\left(-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] + C_2 \exp\left[\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right]$$

On the other hand, the particular solution satisfies

$$Lq_p'' + Rq_p' + \frac{q_p}{C} = 12.$$

Since the inhomogeneous term is a constant, the trial solution will be one as well: $q_p(t) = A$. Plug this into the ODE to determine A.

$$L(A)'' + R(A)' + \frac{A}{C} = 12$$

$$\frac{A}{C} = 12$$

$$A = 12C$$

As a result, the particular solution is $q_p(t) = 12C$, and the general solution is

$$q(t) = C_1 \exp\left[\left(-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] + C_2 \exp\left[\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] + 12C.$$

Differentiate it with respect to t.

$$\begin{aligned} q'(t) = C_1 \left(-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right) \exp\left[\left(-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] \\ + C_2 \left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right) \exp\left[\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] \end{aligned}$$

Now apply the initial conditions, $q(0) = 0$ and $q'(0) = 0$, to determine C_1 and C_2.

$$q(0) = C_1 + C_2 + 12C = 0$$

$$q'(0) = C_1 \left(-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right) + C_2 \left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right) = 0$$

Solving this system of equations yields

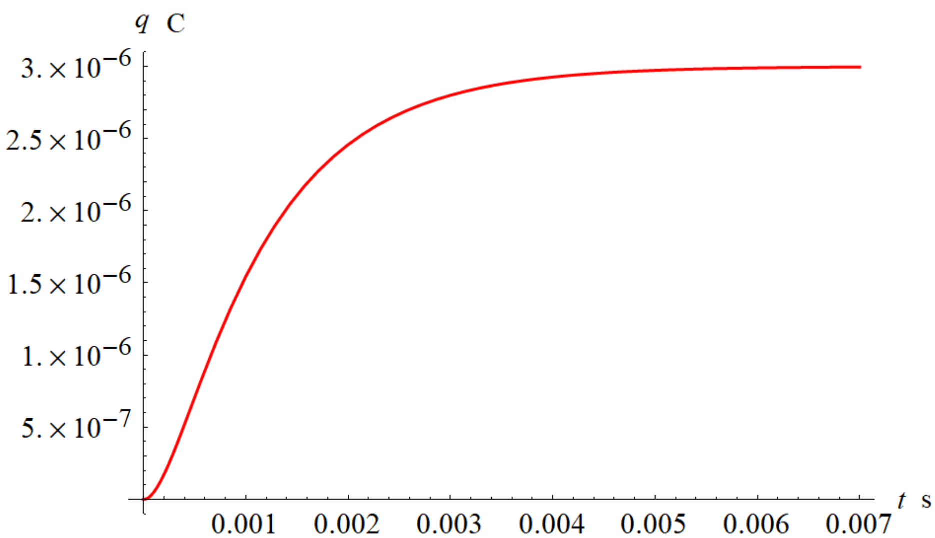
$$C_1 = 6C\left[-1 + R\sqrt{\frac{C}{CR^2 - 4L}}\right] \quad \text{and} \quad C_2 = 6C\left[-1 - R\sqrt{\frac{C}{CR^2 - 4L}}\right].$$

Therefore, the charge on the capacitor (in Coulombs) is

$$q(t) = 6C\left[-1 + R\sqrt{\frac{C}{CR^2 - 4L}}\right]\exp\left[\left(-\frac{R}{2L} - \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] + 6C\left[-1 - R\sqrt{\frac{C}{CR^2 - 4L}}\right]\exp\left[\left(-\frac{R}{2L} + \frac{\sqrt{R^2C^2 - 4LC}}{2LC}\right)t\right] + 12C.$$

Plugging in the numbers, $R = 5 \times 10^3\ \Omega$ and $C = 0.25 \times 10^{-6}$ F and $L = 1$ H, we can plot $q(t)$ versus t and determine the charge at any time we wish.

$$q(t) = 10^{-6}e^{-4000t} - 4 \times 10^{-6}e^{-1000t} + 3 \times 10^{-6}$$



In particular, at $t = 0.001$ s and $t = 0.01$ s the charge on the capacitor is

$$q(0.001) \approx 1.5468 \times 10^{-6}\ \text{C}$$
$$q(0.01) \approx 2.99982 \times 10^{-6}\ \text{C}.$$

Because the coefficient of t in each exponent of e is negative, the exponential functions tend to zero in the limit as $t \to \infty$. Therefore,

$$\lim_{t\to\infty} q(t) = 12C = 3 \times 10^{-6}\ \text{C}.$$