Problem 17

Consider a vibrating system described by the initial value problem

$$u'' + \frac{1}{4}u' + 2u = 2\cos\omega t, \qquad u(0) = 0, \quad u'(0) = 2.$$

(a) Determine the steady state part of the solution of this problem.

(b) Find the amplitude A of the steady state solution in terms of ω.

(c) Plot A versus ω.

(d) Find the maximum value of A and the frequency ω for which it occurs.

Solution

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$u(t) = u_c(t) + u_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_c'' + \frac{1}{4}u_c' + 2u_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u_c = e^{rt}$.

$$u_c = e^{rt} \quad \rightarrow \quad u_c' = re^{rt} \quad \rightarrow \quad u_c'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2e^{rt} + \frac{1}{4}(re^{rt}) + 2(e^{rt}) = 0$$

Divide both sides by $e^{rt}/4$.

$$4r^2 + r + 8 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(4)(8)}}{2(4)} = \frac{-1 \pm \sqrt{-127}}{8} = \frac{-1 \pm i\sqrt{127}}{8} = -\frac{1}{8} \pm i\mu$$

$$r = \left\{-\frac{1}{8} - i\mu, -\frac{1}{8} + i\mu\right\}$$

Two solutions to equation (1) are $u_c = e^{(-1/8 - i\mu)t}$ and $u_c = e^{(-1/8 + i\mu)t}$.

By the principle of superposition, the general solution for u_c is a linear combination of these two.

$$\begin{aligned}
u_c(t) &= C_1e^{(-1/8-i\mu)t} + C_2e^{(-1/8+i\mu)t} \\
&= C_1e^{-t/8-i\mu t} + C_2e^{-t/8+i\mu t} \\
&= C_1e^{-t/8}e^{-i\mu t} + C_2e^{-t/8}e^{i\mu t} \\
&= C_1e^{-t/8}[\cos(-\mu t) + i\sin(-\mu t)] + C_2e^{-t/8}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1e^{-t/8}[\cos(\mu t) - i\sin(\mu t)] + C_2e^{-t/8}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1e^{-t/8}\cos\mu t - iC_1e^{-t/8}\sin\mu t + C_2e^{-t/8}\cos\mu t + iC_2e^{-t/8}\sin\mu t \\
&= (C_1 + C_2)e^{-t/8}\cos\mu t + (-iC_1 + iC_2)e^{-t/8}\sin\mu t \\
&= C_3e^{-t/8}\cos\mu t + C_4e^{-t/8}\sin\mu t \\
&= C_3e^{-t/8}\cos\frac{\sqrt{127}}{8}t + C_4e^{-t/8}\sin\frac{\sqrt{127}}{8}t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$u_p'' + \frac{1}{4}u_p' + 2u_p = 2\cos\omega t.$$

Since there are both even and odd derivatives on the left side, include both sine and cosine in the trial solution to account for the cosine term on the right side: $u_p(t) = B\cos\omega t + D\sin\omega t$. Plug this into the ODE to determine B and D.

$$(B\cos\omega t + D\sin\omega t)'' + \frac{1}{4}(B\cos\omega t + D\sin\omega t)' + 2(B\cos\omega t + D\sin\omega t) = 2\cos\omega t$$

Evaluate the derivatives.

$$(-B\omega^2\cos\omega t - D\omega^2\sin\omega t) + \frac{1}{4}(-B\omega\sin\omega t + D\omega\cos\omega t) + 2(B\cos\omega t + D\sin\omega t) = 2\cos\omega t$$

$$\left(-B\omega^2 + \frac{D\omega}{4} + 2B\right)\cos\omega t + \left(-D\omega^2 - \frac{B\omega}{4} + 2D\right)\sin\omega t = 2\cos\omega t$$

Match the coefficients.

$$\begin{aligned}
-B\omega^2 + \frac{D\omega}{4} + 2B &= 2 \\
-D\omega^2 - \frac{B\omega}{4} + 2D &= 0
\end{aligned}$$

Solving this system of equations yields

$$B = \frac{32(2-\omega^2)}{16\omega^4 - 63\omega^2 + 64} \quad \text{and} \quad D = \frac{8\omega}{16\omega^4 - 63\omega^2 + 64}.$$

Consequently, the particular solution is

$$u_p(t) = \frac{32(2-\omega^2)}{16\omega^4 - 63\omega^2 + 64}\cos\omega t + \frac{8\omega}{16\omega^4 - 63\omega^2 + 64}\sin\omega t,$$

and the general solution is

$$u(t) = C_3e^{-t/8}\cos\frac{\sqrt{127}}{8}t + C_4e^{-t/8}\sin\frac{\sqrt{127}}{8}t + \underbrace{\frac{32(2-\omega^2)}{16\omega^4 - 63\omega^2 + 64}\cos\omega t + \frac{8\omega}{16\omega^4 - 63\omega^2 + 64}\sin\omega t}_{\text{steady state}}.$$

Differentiate it with respect to t.

$$\begin{aligned}u'(t) &= -\frac{C_3}{8}e^{-t/8}\cos\frac{\sqrt{127}}{8}t - C_3\frac{\sqrt{127}}{8}e^{-t/8}\sin\frac{\sqrt{127}}{8}t - \frac{C_4}{8}e^{-t/8}\sin\frac{\sqrt{127}}{8}t \\ &\quad + C_4\frac{\sqrt{127}}{8}e^{-t/8}\cos\frac{\sqrt{127}}{8}t - \omega\frac{32(2-\omega^2)}{16\omega^4-63\omega^2+64}\sin\omega t + \omega\frac{8\omega}{16\omega^4-63\omega^2+64}\cos\omega t\end{aligned}$$

Now apply the initial conditions to determine C_3 and C_4.

$$u(0) = C_3 + \frac{32(2-\omega^2)}{16\omega^4-63\omega^2+64} = 0$$

$$u'(0) = -\frac{C_3}{8} + C_4\frac{\sqrt{127}}{8} + \omega\frac{8\omega}{16\omega^4-63\omega^2+64} = 2$$

Solving this system of equations yields

$$C_3 = \frac{32(\omega^2-2)}{16\omega^4-63\omega^2+64} \quad \text{and} \quad C_4 = \frac{16(16\omega^4-65\omega^2+60)}{\sqrt{127}(16\omega^4-63\omega^2+64)}.$$

Therefore,

$$\begin{aligned}u(t) &= \frac{32(\omega^2-2)}{16\omega^4-63\omega^2+64}e^{-t/8}\cos\frac{\sqrt{127}}{8}t + \frac{16(16\omega^4-65\omega^2+60)}{\sqrt{127}(16\omega^4-63\omega^2+64)}e^{-t/8}\sin\frac{\sqrt{127}}{8}t \\ &\quad + \underbrace{\frac{32(2-\omega^2)}{16\omega^4-63\omega^2+64}\cos\omega t + \frac{8\omega}{16\omega^4-63\omega^2+64}\sin\omega t.}_{\text{steady state}}\end{aligned}$$

Introduce an amplitude A and a phase δ to combine the two sinusoidal terms of the steady state into one.

$$\begin{aligned}u(t) &= \frac{32(\omega^2-2)}{16\omega^4-63\omega^2+64}e^{-t/8}\cos\frac{\sqrt{127}}{8}t + \frac{16(16\omega^4-65\omega^2+60)}{\sqrt{127}(16\omega^4-63\omega^2+64)}e^{-t/8}\sin\frac{\sqrt{127}}{8}t \\ &\quad + A\cos\delta\cos\omega t + A\sin\delta\sin\omega t \\ &= \frac{32(\omega^2-2)}{16\omega^4-63\omega^2+64}e^{-t/8}\cos\frac{\sqrt{127}}{8}t + \frac{16(16\omega^4-65\omega^2+60)}{\sqrt{127}(16\omega^4-63\omega^2+64)}e^{-t/8}\sin\frac{\sqrt{127}}{8}t \\ &\quad + A\cos(\omega t-\delta)\end{aligned}$$

A and δ satisfy the following system of equations.

$$A\cos\delta = \frac{32(2-\omega^2)}{16\omega^4-63\omega^2+64}$$

$$A\sin\delta = \frac{8\omega}{16\omega^4-63\omega^2+64}$$

To find A, square both sides of each equation

$$A^2\cos^2\delta = \left[\frac{32(2-\omega^2)}{16\omega^4-63\omega^2+64}\right]^2$$

$$A^2\sin^2\delta = \left[\frac{8\omega}{16\omega^4-63\omega^2+64}\right]^2$$

and then add the respective sides together.

$$A^2\cos^2\delta + A^2\sin^2\delta = \left[\frac{32(2-\omega^2)}{16\omega^4 - 63\omega^2 + 64}\right]^2 + \left[\frac{8\omega}{16\omega^4 - 63\omega^2 + 64}\right]^2$$

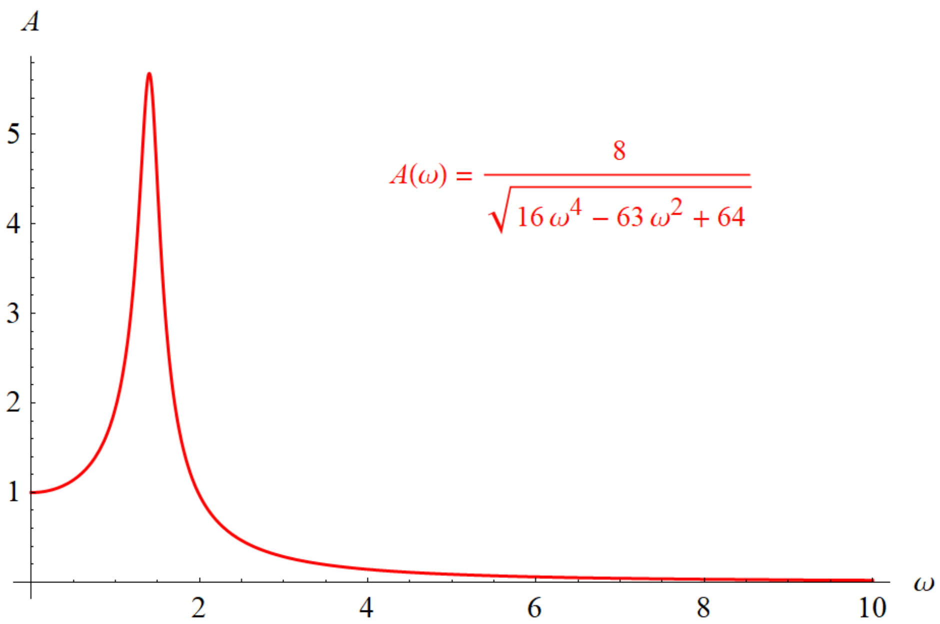
Simplify both sides.

$$A^2(\cos^2\delta + \sin^2\delta) = \frac{64}{16\omega^4 - 63\omega^2 + 64}$$

$$A^2 = \frac{64}{16\omega^4 - 63\omega^2 + 64}$$

$$A = \frac{8}{\sqrt{16\omega^4 - 63\omega^2 + 64}}$$

Below is a plot of A versus ω.



In order to find the value of ω that maximizes A, differentiate A with respect to ω

$$A'(\omega) = -\frac{4}{(16\omega^4 - 63\omega^2 + 64)^{3/2}}(64\omega^3 - 126\omega)$$

and then set it equal to zero.

$$-\frac{4}{(16\omega^4 - 63\omega^2 + 64)^{3/2}}(64\omega^3 - 126\omega) = 0$$

Solve for ω.

$$64\omega^3 - 126\omega = 0$$

$$2\omega(32\omega^2 - 63) = 0$$

Use the zero product theorem.

$$2\omega = 0 \quad \text{or} \quad 32\omega^2 - 63 = 0$$

$$\omega = 0 \quad \text{or} \quad \omega^2 = \frac{63}{32}$$

$$\omega = \pm\sqrt{\frac{63}{32}}$$

Since ω has to be positive,

$$\omega_{\text{max}} = \sqrt{\frac{63}{32}} \approx 1.403.$$

The amplitude that corresponds with this value of ω is

$$\begin{aligned} A_{\text{max}} = A(\omega_{\text{max}}) &= \frac{8}{\sqrt{16\omega_{\text{max}}^4 - 63\omega_{\text{max}}^2 + 64}} \\ &= \frac{8}{\sqrt{16\left(\sqrt{\frac{63}{32}}\right)^4 - 63\left(\sqrt{\frac{63}{32}}\right)^2 + 64}} \\ &= \frac{64}{\sqrt{127}} \\ &\approx 5.679. \end{aligned}$$