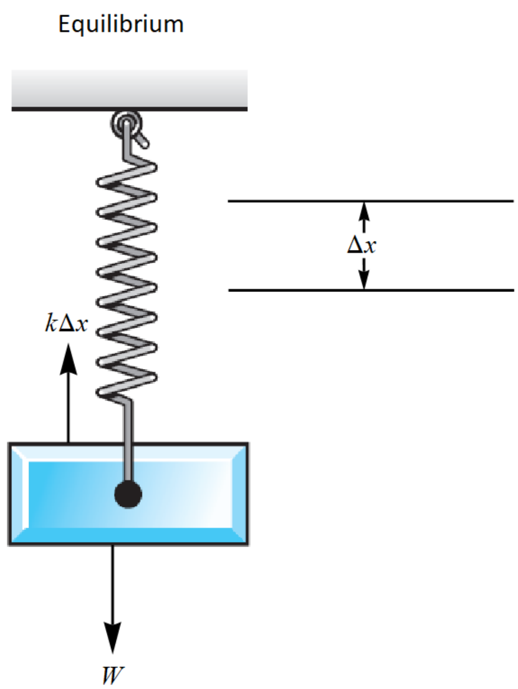
Problem 10

A mass that weighs 8 lb stretches a spring 6 in. The system is acted on by an external force of $8 \sin 8t$ lb. If the mass is pulled down 3 in and then released, determine the position of the mass at any time. Determine the first four times at which the velocity of the mass is zero.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it in equilibrium are due to the spring and gravity.



The gravitational and spring forces balance each other.

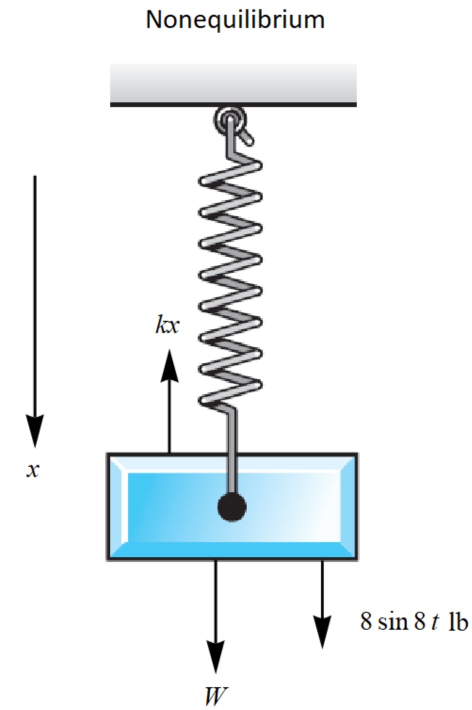
$$W = k\Delta x$$

From this equation, k can be determined.

$$8 \text{ lb} = k(6 \text{ in})$$

$$k = \frac{4}{3}\,\frac{\text{lb}}{\cancel{\text{in}}} \times \frac{12\,\cancel{\text{in}}}{1 \text{ ft}} = 16\,\frac{\text{lb}}{\text{ft}}$$

Consider now the mass in nonequilibrium.



Apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-kx + W + 8\sin 8t = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-kx + W + 8\sin 8t = mx''$$

$$mx'' + kx = W + 8\sin 8t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2 e^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + k = 0$$
$$r^2 = -\frac{k}{m}$$
$$r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

Two solutions to equation (1) are then

$$x_c = e^{-i\omega t} \quad \text{and} \quad x_c = e^{i\omega t}.$$

By the principle of superposition, the general solution for x_c is a linear combination of these two.

$$\begin{aligned}
x_c(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\
&= C_1[\cos(-\omega t) + i\sin(-\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\
&= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\
&= C_1 \cos\omega t - iC_1 \sin\omega t + C_2 \cos\omega t + iC_2 \sin\omega t \\
&= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\
&= C_3 \cos\omega t + C_4 \sin\omega t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$mx_p'' + kx_p = W + 8\sin 8t.$$

For the first term on the right side, we'll include a constant A in the trial solution. Also, since there are only even derivatives on the left side, we'll include $B\sin 8t$ in the trial solution to account for the second term on the right side. Substitute $x_p(t) = A + B\sin 8t$ into the equation to determine A and B.

$$m(A + B\sin 8t)'' + k(A + B\sin 8t) = W + 8\sin 8t$$
$$m(-64B\sin 3t) + kA + kB\sin 8t = W + 8\sin 8t$$
$$kA + (kB - 64mB)\sin 8t = W + 8\sin 8t$$

Matching the coefficients, we have

$$kA = W$$
$$kB - 64mB = 8.$$

Solving this system yields

$$A = \frac{W}{k} \quad \text{and} \quad B = \frac{8}{k - 64m}.$$

The particular solution is then

$$x_p(t) = \frac{W}{k} + \frac{8}{k - 64m}\sin 8t,$$

which means the general solution is

$$\begin{aligned}
x(t) &= C_3 \cos\omega t + C_4 \sin\omega t + \frac{W}{k} + \frac{8}{k - 64m}\sin 8t \\
&= C_3 \cos\sqrt{\frac{k}{m}}t + C_4 \sin\sqrt{\frac{k}{m}}t + \frac{W}{k} + \frac{8}{k - 64m}\sin 8t.
\end{aligned}$$

Take a derivative of it with respect to t.

$$x'(t) = -C_3\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t + C_4\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t + \frac{64}{k-64m}\cos 8t$$

Now apply the initial conditions,

$$\begin{aligned} x(0) &= 6\text{ in} + 3\text{ in} = 9\ \cancel{\text{in}} \times \frac{1\text{ ft}}{12\ \cancel{\text{in}}} = \frac{3}{4}\text{ ft} \\ x'(0) &= 0\ \frac{\text{ft}}{\text{s}}, \end{aligned}$$

to determine C_3 and C_4.

$$\begin{aligned} x(0) &= C_3 + \frac{W}{k} = \frac{3}{4} \\ x'(0) &= C_4\sqrt{\frac{k}{m}} + \frac{64}{k-64m} = 0 \end{aligned}$$

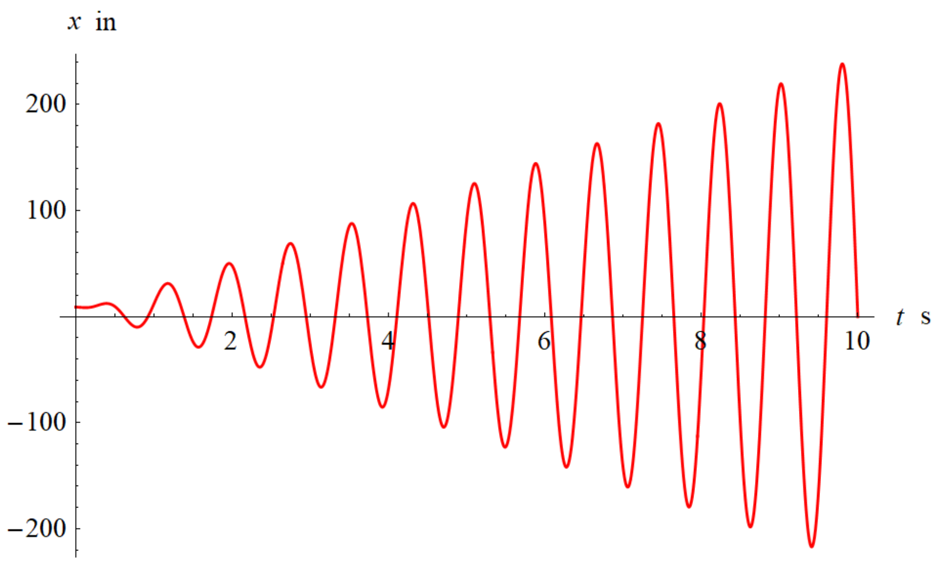
Solving this system of equations yields

$$C_3 = \frac{3}{4} - \frac{W}{k} = \frac{1}{4} \quad \text{and} \quad C_4 = -\frac{64}{k-64m}\sqrt{\frac{m}{k}},$$

so

$$x(t) = \frac{1}{4}\cos\sqrt{\frac{k}{m}}t - \frac{64}{k-64m}\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t + \frac{W}{k} + \frac{8}{k-64m}\sin 8t.$$

Finally, plug in the numbers, $W = 8$ lb and $k = 16$ lb/ft. The mass m is obtained by dividing the weight by the gravity: $m = W/g = 8\text{ lb}/(32.2\text{ ft/s}^2)$. As $x(t)$ is in feet, multiply the result by 12 to convert it to inches.



Zoom in the graph to the first few oscillations. The points where the slope is zero are marked approximately.