Problem 11

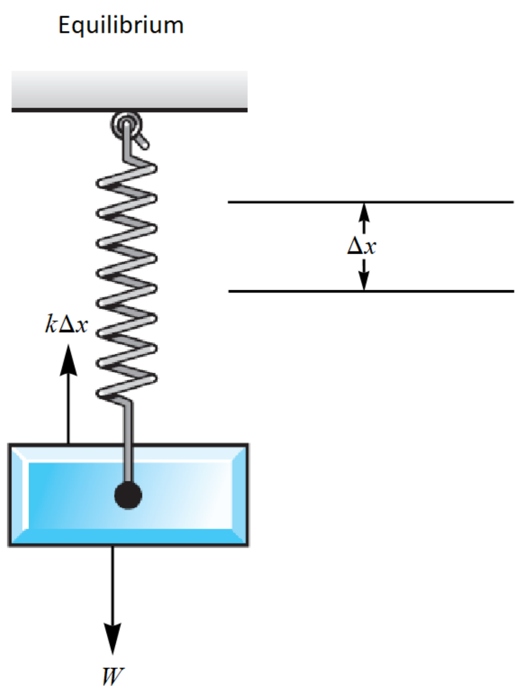
A spring is stretched 6 in by a mass that weighs 8 lb. The mass is attached to a dashpot mechanism that has a damping constant of 0.25 lb $\cdot$ s/ft and is acted on by an external force of $4\cos 2t$ lb.

(a) Determine the steady state response of this system.

(b) If the given mass is replaced by a mass m, determine the value of m for which the amplitude of the steady state response is maximum.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it in equilibrium are due to the spring and gravity.



The gravitational and spring forces balance each other.

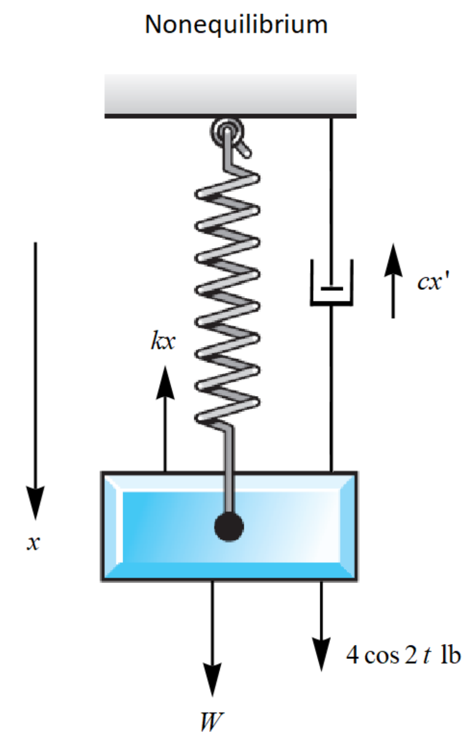
$$W = k\Delta x$$

From this equation, k can be determined.

$$8 \text{ lb} = k(6 \text{ in})$$

$$k = \frac{4}{3} \frac{\text{lb}}{\cancel{\text{in}}} \times \frac{12 \cancel{\text{in}}}{1 \text{ ft}} = 16 \frac{\text{lb}}{\text{ft}}$$

Consider now the mass in nonequilibrium. From now on, $c = 0.25$ lb $\cdot$ s/ft, $k = 16$ lb/ft, $W = 8$ lb, and $m = (8 \text{ lb})/(32.2 \text{ ft/s}^2)$.



Apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-cx' - kx + W + 4\cos 2t = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-cx' - kx + W + 4\cos 2t = mx''$$

$$mx'' + cx' + kx = W + 4\cos 2t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + cx_c' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2 e^{rt}) + c(re^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + cr + k = 0$$

In this problem $c^2 - 4km < 0$. Make it so that the quantity under the square root is positive.

$$r = \frac{-c \pm \sqrt{c^2 - 4km}}{2m} = \frac{-c \pm i\sqrt{4km - c^2}}{2m} = -\frac{c}{2m} \pm i\frac{\sqrt{4km - c^2}}{2m} = -\frac{c}{2m} \pm i\mu$$

Two solutions to equation (1) are then

$$x_c = e^{(-c/2m - i\mu)t} \quad \text{and} \quad x_c = e^{(-c/2m + i\mu)t}.$$

By the principle of superposition, the general solution for x_c is a linear combination of these two.

$$\begin{aligned}
x_c(t) &= C_1 e^{(-c/2m - i\mu)t} + C_2 e^{(-c/2m + i\mu)t} \\
&= C_1 e^{-ct/2m - i\mu t} + C_2 e^{-ct/2m + i\mu t} \\
&= C_1 e^{-ct/2m} e^{-i\mu t} + C_2 e^{-ct/2m} e^{i\mu t} \\
&= C_1 e^{-ct/2m}[\cos(-\mu t) + i\sin(-\mu t)] + C_2 e^{-ct/2m}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-ct/2m}[\cos(\mu t) - i\sin(\mu t)] + C_2 e^{-ct/2m}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-ct/2m}\cos\mu t - iC_1 e^{-ct/2m}\sin\mu t + C_2 e^{-ct/2m}\cos\mu t + iC_2 e^{-ct/2m}\sin\mu t \\
&= (C_1 + C_2)e^{-ct/2m}\cos\mu t + (-iC_1 + iC_2)e^{-ct/2m}\sin\mu t \\
&= C_3 e^{-ct/2m}\cos\mu t + C_4 e^{-ct/2m}\sin\mu t \\
&= e^{-ct/2m}(C_3\cos\mu t + C_4\sin\mu t)
\end{aligned}$$

On the other hand, the particular solution satisfies

$$mx_p'' + cx_p' + kx_p = W + 4\cos 2t.$$

For the first term on the right side, we'll include a constant A in the trial solution. Also, since there are even and odd derivatives on the left side, we'll include $B\cos 2t + C\sin 2t$ in the trial solution to account for the second term on the right side. Substitute $x_p(t) = A + B\cos 2t + C\sin 2t$ into the equation to determine A and B and C.

$$m(A + B\cos 2t + C\sin 2t)'' + c(A + B\cos 2t + C\sin 2t)' + k(A + B\cos 2t + C\sin 2t) = W + 4\cos 2t$$

$$m(-4B\cos 2t - 4C\sin 2t) + c(-2B\sin 2t + 2C\cos 2t) + k(A + B\cos 2t + C\sin 2t) = W + 4\cos 2t$$

$$kA + [2cC + B(k - 4m)]\cos 2t + [-2cB + (k - 4m)C]\sin 2t = W + 4\cos 2t$$

Matching the coefficients, we have

$$\begin{aligned}
kA &= W \\
2cC + (k - 4m)B &= 4 \\
-2cB + (k - 4m)C &= 0.
\end{aligned}$$

Solving this system yields

$$A = \frac{W}{k} \quad \text{and} \quad B = \frac{4(k - 4m)}{4c^2 + (k - 4m)^2} \quad \text{and} \quad C = \frac{8c}{4c^2 + (k - 4m)^2}.$$

The particular solution is then

$$x_p(t) = \frac{W}{k} + \frac{4(k-4m)}{4c^2 + (k-4m)^2}\cos 2t + \frac{8c}{4c^2 + (k-4m)^2}\sin 2t,$$

which means the general solution is

$$\begin{aligned} x(t) &= e^{-ct/2m}(C_3\cos\mu t + C_4\sin\mu t) + \frac{W}{k} + \frac{4(k-4m)}{4c^2 + (k-4m)^2}\cos 2t + \frac{8c}{4c^2 + (k-4m)^2}\sin 2t \\ &= e^{-ct/2m}\left(C_3\cos\frac{\sqrt{4km-c^2}}{2m}t + C_4\sin\frac{\sqrt{4km-c^2}}{2m}t\right) + \underbrace{\frac{W}{k} + \frac{4(k-4m)}{4c^2 + (k-4m)^2}\cos 2t + \frac{8c}{4c^2 + (k-4m)^2}\sin 2t}_{\text{steady state}}. \end{aligned}$$

Introduce an amplitude R and a phase δ to combine the two sinusoidal terms.

$$\begin{aligned} x(t) &= e^{-ct/2m}\left(C_3\cos\frac{\sqrt{4km-c^2}}{2m}t + C_4\sin\frac{\sqrt{4km-c^2}}{2m}t\right) + \frac{W}{k} + R\cos\delta\cos 2t + R\sin\delta\sin 2t \\ &= e^{-ct/2m}\left(C_3\cos\frac{\sqrt{4km-c^2}}{2m}t + C_4\sin\frac{\sqrt{4km-c^2}}{2m}t\right) + \frac{W}{k} + R\cos(2t-\delta) \end{aligned}$$

R and δ satisfy the following system of equations.

$$\begin{aligned} R\cos\delta &= \frac{4(k-4m)}{4c^2 + (k-4m)^2} \\ R\sin\delta &= \frac{8c}{4c^2 + (k-4m)^2} \end{aligned}$$

Square both sides of each equation and then add the respective sides to find R.

$$\begin{aligned} R &= \sqrt{\left[\frac{4(k-4m)}{4c^2 + (k-4m)^2}\right]^2 + \left[\frac{8c}{4c^2 + (k-4m)^2}\right]^2} \\ &= \frac{4}{\sqrt{4c^2 + (k-4m)^2}} \end{aligned}$$

To find the value of m that maximizes R, differentiate R with respect to m

$$R'(m) = -\frac{2}{[4c^2 + (k-4m)^2]^{3/2}}[2(k-4m)](-4)$$

and then set it equal to zero.

$$-\frac{2}{[4c^2 + (k-4m)^2]^{3/2}}[2(k-4m)](-4) = 0$$

Solving for m yields

$$k - 4m = 0 \quad \rightarrow \quad m = \frac{k}{4} = 4\,\frac{\text{lb}\cdot\text{s}^2}{\text{ft}}.$$