Problem 22

Problems 21 through 23 deal with the initial value problem

$$u'' + 0.125u' + 4u = F(t), \qquad u(0) = 2, \quad u'(0) = 0.$$

In each of these problems:

(a) Plot the given forcing function $F(t)$ versus t, and also plot the solution $u(t)$ versus t on the same set of axes. Use a t interval that is long enough so the initial transients are substantially eliminated. Observe the relation between the amplitude and phase of the forcing term and the amplitude and phase of the response. Note that $\omega_0 = \sqrt{k/m} = 2$.

(b) Draw the phase plot of the solution; that is, plot u' versus u.

$$F(t) = 3\cos 2t$$

Solution

$$u'' + \frac{1}{8}u' + 4u = 3\cos 2t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$u(t) = u_c(t) + u_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$u_c'' + \frac{1}{8}u_c' + 4u_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $u_c = e^{rt}$.

$$u_c = e^{rt} \quad \rightarrow \quad u_c' = re^{rt} \quad \rightarrow \quad u_c'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^2e^{rt} + \frac{1}{8}(re^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by $e^{rt}/8$.

$$8r^2 + r + 32 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4(8)(32)}}{2(8)} = \frac{-1 \pm \sqrt{-1023}}{16} = \frac{-1 \pm i\sqrt{1023}}{16} = -\frac{1}{16} \pm i\mu$$

$$r = \left\{-\frac{1}{16} - i\mu, -\frac{1}{16} + i\mu\right\}$$

Two solutions to equation (1) are $u_c = e^{(-1/16-i\mu)t}$ and $u_c = e^{(-1/16+i\mu)t}$. By the principle of superposition, the general solution for u_c is a linear combination of these two.

$$\begin{aligned}
u_c(t) &= C_1 e^{(-1/16-i\mu)t} + C_2 e^{(-1/16+i\mu)t} \\
&= C_1 e^{-t/16-i\mu t} + C_2 e^{-t/16+i\mu t} \\
&= C_1 e^{-t/16} e^{-i\mu t} + C_2 e^{-t/16} e^{i\mu t} \\
&= C_1 e^{-t/16}[\cos(-\mu t) + i\sin(-\mu t)] + C_2 e^{-t/16}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-t/16}[\cos(\mu t) - i\sin(\mu t)] + C_2 e^{-t/16}[\cos(\mu t) + i\sin(\mu t)] \\
&= C_1 e^{-t/16}\cos\mu t - iC_1 e^{-t/16}\sin\mu t + C_2 e^{-t/16}\cos\mu t + iC_2 e^{-t/16}\sin\mu t \\
&= (C_1 + C_2)e^{-t/16}\cos\mu t + (-iC_1 + iC_2)e^{-t/16}\sin\mu t \\
&= C_3 e^{-t/16}\cos\mu t + C_4 e^{-t/16}\sin\mu t \\
&= e^{-t/16}\left(C_3\cos\frac{\sqrt{1023}}{16}t + C_4\sin\frac{\sqrt{1023}}{16}t\right)
\end{aligned}$$

On the other hand, the particular solution satisfies

$$u_p'' + \frac{1}{8}u_p' + 4u_p = 3\cos 2t.$$

Since there are both odd and even derivatives on the left side, include sine and cosine in the trial solution to account for the cosine term on the right side: $u_p(t) = A\cos 2t + B\sin 2t$. Plug this into the ODE to determine A and B.

$$(A\cos 2t + B\sin 2t)'' + \frac{1}{8}(A\cos 2t + B\sin 2t)' + 4(A\cos 2t + B\sin 2t) = 3\cos 2t$$

Evaluate the derivatives.

$$(-4A\cos 2t - 4B\sin 2t) + \frac{1}{8}(-2A\sin 2t + 2B\cos 2t) + 4(A\cos 2t + B\sin 2t) = 3\cos 2t$$

$$\left(-4A + \frac{1}{4}B + 4A\right)\cos 2t + \left(-4B - \frac{1}{4}A + 4B\right)\sin 2t = 3\cos 2t$$

Match the coefficients.

$$\begin{aligned}
-4A + \frac{1}{4}B + 4A &= 3 \\
-4B - \frac{1}{4}A + 4B &= 0
\end{aligned}$$

Solving this system of equations for A and B yields

$$A = 0 \quad \text{and} \quad B = 12.$$

Consequently, the particular solution is

$$u_p(t) = 12\sin 2t,$$

and the general solution is

$$u(t) = e^{-t/16}\left(C_3\cos\frac{\sqrt{1023}}{16}t + C_4\sin\frac{\sqrt{1023}}{16}t\right) + 12\sin 2t.$$

Differentiate it with respect to t.

$$u'(t) = -\frac{1}{16}e^{-t/16}\left(C_3\cos\frac{\sqrt{1023}}{16}t + C_4\sin\frac{\sqrt{1023}}{16}t\right) + e^{-t/16}\left(-C_3\frac{\sqrt{1023}}{16}\sin\frac{\sqrt{1023}}{16}t + C_4\frac{\sqrt{1023}}{16}\cos\frac{\sqrt{1023}}{16}t\right) + 24\cos 2t$$

Now apply the initial conditions to determine C_3 and C_4.

$$u(0) = C_3 = 2$$

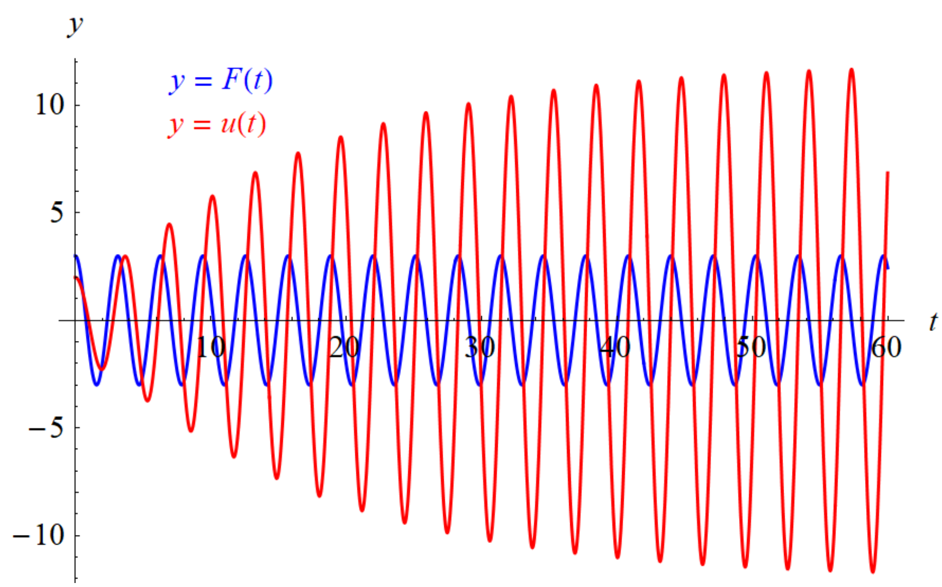
$$u'(0) = -\frac{1}{16}C_3 + C_4\frac{\sqrt{1023}}{16} + 24 = 0$$

Solving this system of equations yields

$$C_3 = 2 \quad \text{and} \quad C_4 = -\frac{382}{\sqrt{1023}}.$$

Therefore,

$$u(t) = e^{-t/16}\left(2\cos\frac{\sqrt{1023}}{16}t - \frac{382}{\sqrt{1023}}\sin\frac{\sqrt{1023}}{16}t\right) + 12\sin 2t.$$



The forcing function lags behind the response by $\pi/2$ radians. Additionally, the amplitude of the steady-state response is six times that of the forcing function.

Below is a phase plot for $0 \le t \le 20$.