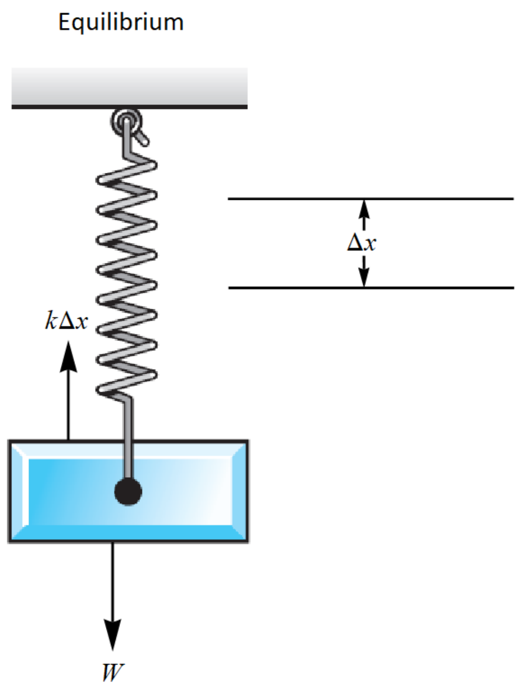
Problem 5

A mass weighing 4 lb stretches a spring 1.5 in. The mass is given a positive displacement of 2 in from its equilibrium position and released with no initial velocity. Assuming that there is no damping and that the mass is acted on by an external force of $2\cos 3t$ lb, formulate the initial value problem describing the motion of the mass.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it in equilibrium are due to the spring and gravity.



The gravitational and spring forces balance each other.

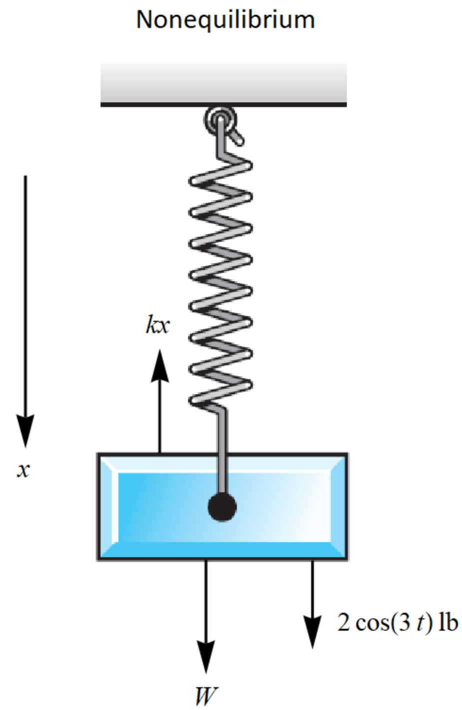
$$W = k\Delta x$$

From this equation, k can be determined.

$$4 \text{ lb} = k(1.5 \text{ in})$$

$$k = \frac{8}{3}\,\frac{\text{lb}}{\cancel{\text{in}}} \times \frac{12\,\cancel{\text{in}}}{1 \text{ ft}} = 32\,\frac{\text{lb}}{\text{ft}}$$

Consider now the mass in nonequilibrium.



Apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-kx + W + 2\cos 3t = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-kx + W + 2\cos 3t = mx''$$

$$mx'' + kx = W + 2\cos 3t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2e^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + k = 0$$
$$r^2 = -\frac{k}{m}$$
$$r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

Two solutions to equation (1) are then

$$x_c = e^{-i\omega t} \quad \text{and} \quad x_c = e^{i\omega t}.$$

By the principle of superposition, the general solution for x_c is a linear combination of these two.

$$\begin{aligned}
x_c(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\
&= C_1[\cos(-\omega t) + i\sin(-\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\
&= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\
&= C_1\cos\omega t - iC_1\sin\omega t + C_2\cos\omega t + iC_2\sin\omega t \\
&= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\
&= C_3\cos\omega t + C_4\sin\omega t
\end{aligned}$$

On the other hand, the particular solution satisfies

$$mx_p'' + kx_p = W + 2\cos 3t.$$

For the first term on the right side, we'll include a constant A in the trial solution. Also, since there are only even derivatives on the left side, we'll include $B\cos 3t$ in the trial solution to account for the second term on the right side. Substitute $x_p(t) = A + B\cos 3t$ into the equation to determine A and B.

$$m(A + B\cos 3t)'' + k(A + B\cos 3t) = W + 2\cos 3t$$
$$m(-9B\cos 3t) + kA + kB\cos 3t = W + 2\cos 3t$$
$$kA + (kB - 9mB)\cos 3t = W + 2\cos 3t$$

Matching the coefficients, we have

$$\begin{aligned}
kA &= W \\
kB - 9mB &= 2.
\end{aligned}$$

Solving this system yields

$$A = \frac{W}{k} \quad \text{and} \quad B = \frac{2}{k - 9m}.$$

The particular solution is then

$$x_p(t) = \frac{W}{k} + \frac{2}{k - 9m}\cos 3t,$$

which means the general solution is

$$\begin{aligned}
x(t) &= C_3\cos\omega t + C_4\sin\omega t + \frac{W}{k} + \frac{2}{k - 9m}\cos 3t \\
&= C_3\cos\sqrt{\frac{k}{m}}t + C_4\sin\sqrt{\frac{k}{m}}t + \frac{W}{k} + \frac{2}{k - 9m}\cos 3t.
\end{aligned}$$

Take a derivative of it with respect to t.

$$x'(t) = -C_3\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t + C_4\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t - \frac{6}{k-9m}\sin 3t$$

Now apply the initial conditions,

$$\begin{aligned} x(0) &= 1.5 \text{ in} + 2 \text{ in} = 3.5 \cancel{\text{in}} \times \frac{1 \text{ ft}}{12 \cancel{\text{in}}} = \frac{7}{24} \text{ ft} \\ x'(0) &= 0 \ \frac{\text{ft}}{\text{s}}, \end{aligned}$$

to determine C_3 and C_4.

$$\begin{aligned} x(0) &= C_3 + \frac{W}{k} + \frac{2}{k-9m} = \frac{7}{24} \\ x'(0) &= C_4\sqrt{\frac{k}{m}} = 0 \end{aligned}$$

Solving this system of equations yields

$$C_3 = \frac{7}{24} - \frac{W}{k} - \frac{2}{k-9m} \quad \text{and} \quad C_4 = 0,$$

so

$$x(t) = \left(\frac{7}{24} - \frac{W}{k} - \frac{2}{k-9m}\right)\cos\sqrt{\frac{k}{m}}t + \frac{W}{k} + \frac{2}{k-9m}\cos 3t.$$

Finally, plug in the numbers, $W = 4$ lb and $k = 32$ lb/ft. The mass m is obtained by dividing the weight by the gravity: $m = W/g = 4 \text{ lb}/(32.2 \text{ ft/s}^2)$. As $x(t)$ is in feet, multiply the result by 12 to convert it to inches.