# Problem 8

(a) Find the solution of the initial value problem in Problem 6.

(b) Identify the transient and steady state parts of the solution.

(c) Plot the graph of the steady state solution.

(d) If the given external force is replaced by a force of $2\cos\omega t$ of frequency $\omega$, find the value of $\omega$ for which the amplitude of the forced response is maximum.

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**Solution**

The initial value problem in Problem 6 was

$$mx'' + cx' + kx = mg + 10\sin\frac{t}{2}, \qquad x(0) = \frac{1}{10}, \quad x'(0) = \frac{3}{100},$$

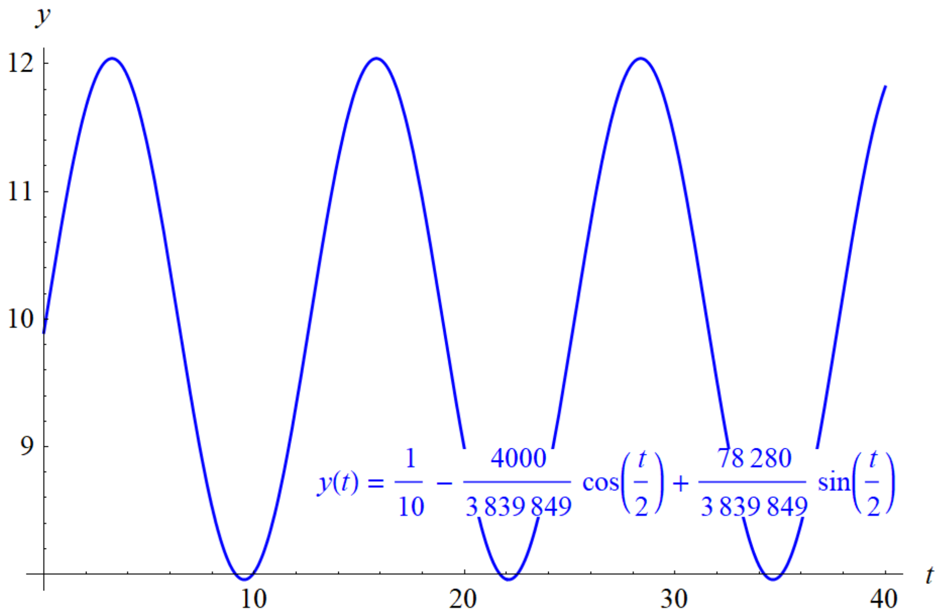
where

$$\begin{aligned} m &= 5 \text{ kg} \\ c &= 50 \ \frac{\text{N}\cdot\text{s}}{\text{m}} \\ k &= 490.5 \ \frac{\text{N}}{\text{m}} \\ g &= 9.81 \ \frac{\text{m}}{\text{s}^2}, \end{aligned}$$

and its solution was found to be

$$x(t) = \overbrace{e^{-5t}\left(\frac{4000}{3\,839\,849}\cos\sqrt{73.1}t + \frac{9\,605\,547}{383\,984\,900\sqrt{73.1}}\sin\sqrt{73.1}t\right)}^{\text{transient component}} + \underbrace{\frac{1}{10} - \frac{4000}{3\,839\,849}\cos\frac{t}{2} + \frac{78\,280}{3\,839\,849}\sin\frac{t}{2}}_{\text{steady component}}.$$

A plot of the steady-state solution is given below.




Suppose now that the external force is $2\cos\omega t$ rather than $10\sin(t/2)$. The equation of motion is then

$$mx'' + cx' + kx = mg + 2\cos\omega t.$$

Determine a particular solution by plugging in the trial solution,

$$x_p(t) = A + B\cos\omega t + C\sin\omega t,$$

to the ODE.

$$m(A+B\cos\omega t+C\sin\omega t)''+c(A+B\cos\omega t+C\sin\omega t)'+k(A+B\cos\omega t+C\sin\omega t) = mg+2\cos\omega t$$

$$m(-B\omega^2\cos\omega t-C\omega^2\sin\omega t)+c(-B\omega\sin\omega t+C\omega\cos\omega t)+k(A+B\cos\omega t+C\sin\omega t) = mg+2\cos\omega t$$

$$kA + [\omega cC + B(k - m\omega^2)]\cos\omega t + (kC - c\omega B - m\omega^2 C)\sin\omega t = mg + 2\cos\omega t$$

Match the coefficients.

$$\begin{aligned} kA &= mg \\ \omega cC + B(k - m\omega^2) &= 2 \\ kC - c\omega B - m\omega^2 C &= 0 \end{aligned}$$

Solving this system of equations yields

$$A = \frac{mg}{k} \quad \text{and} \quad B = \frac{2(k - m\omega^2)}{c^2\omega^2 + (k - m\omega^2)^2} \quad \text{and} \quad C = \frac{2c\omega}{c^2\omega^2 + (k - m\omega^2)^2},$$

which means the particular solution is

$$x_p(t) = \frac{mg}{k} + \frac{2(k - m\omega^2)}{c^2\omega^2 + (k - m\omega^2)^2}\cos\omega t + \frac{2c\omega}{c^2\omega^2 + (k - m\omega^2)^2}\sin\omega t.$$

Introduce an amplitude $R$ and a phase $\delta$ to combine the two sinusoidal terms.

$$\begin{aligned} x_p(t) &= \frac{mg}{k} + R\cos\delta\cos\omega t + R\sin\delta\sin\omega t \\ &= \frac{mg}{k} + R\cos(\omega t - \delta) \end{aligned}$$

$R$ and $\delta$ satisfy the following system of equations.

$$\begin{aligned} R\cos\delta &= \frac{2(k - m\omega^2)}{c^2\omega^2 + (k - m\omega^2)^2} \\ R\sin\delta &= \frac{2c\omega}{c^2\omega^2 + (k - m\omega^2)^2} \end{aligned}$$

Determine $R$ by squaring both sides of each equation and then adding the respective sides.

$$\begin{aligned} R &= \sqrt{\left[\frac{2(k - m\omega^2)}{c^2\omega^2 + (k - m\omega^2)^2}\right]^2 + \left[\frac{2c\omega}{c^2\omega^2 + (k - m\omega^2)^2}\right]^2} \\ &= \frac{2}{\sqrt{c^2\omega^2 + (k - m\omega^2)^2}} \end{aligned}$$

To find the value of $\omega$ that maximizes $R$, differentiate $R$ with respect to $\omega$

$$R'(\omega) = -\frac{1}{[c^2\omega^2 + (k - m\omega^2)^2]^{3/2}}[2c^2\omega + 2(k - m\omega^2)\cdot(-2m\omega)]$$

and then set it equal to zero.

$$-\frac{1}{[c^2\omega^2 + (k - m\omega^2)^2]^{3/2}}[2c^2\omega + 2(k - m\omega^2)\cdot(-2m\omega)] = 0$$

$$\begin{aligned} 2c^2\omega + 2(k - m\omega^2)\cdot(-2m\omega) &= 0 \\ 2\omega[c^2 - 2m(k - m\omega^2)] &= 0 \end{aligned}$$

Use the zero product theorem.

$$\begin{aligned} 2\omega = 0 \quad &\text{or} \quad c^2 - 2m(k - m\omega^2) = 0 \\ \omega = 0 \quad &\text{or} \quad \omega = \pm\frac{\sqrt{2km - c^2}}{\sqrt{2}m} \end{aligned}$$

Since we're looking for a positive value of $\omega$, we choose

$$\omega = \frac{\sqrt{2km - c^2}}{\sqrt{2}m} \approx 6.935\ \frac{\text{rad}}{\text{s}}.$$