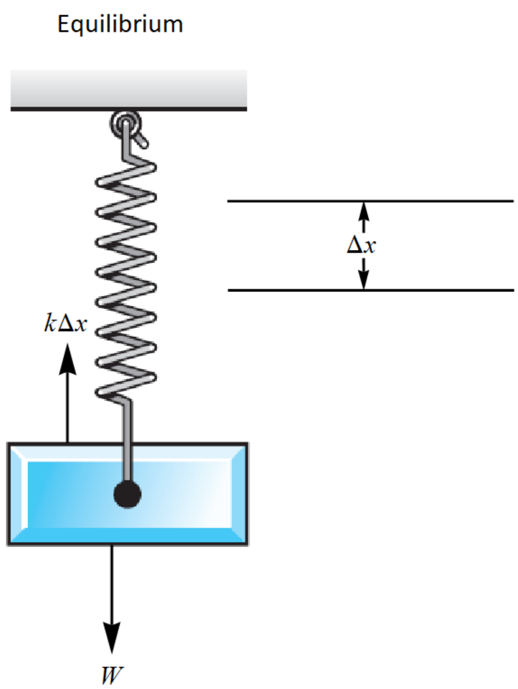
Problem 9

If an undamped spring-mass system with a mass that weighs 6 lb and a spring constant 1 lb/in is suddenly set in motion at $t = 0$ by an external force of $4\cos 7t$ lb, determine the position of the mass at any time, and draw a graph of the displacement versus t.

Solution

Start by drawing a free-body diagram of the mass. The two forces acting on it in equilibrium are due to the spring and gravity.



The gravitational and spring forces balance each other.

$$W = k\Delta x$$

From this equation, the equilibrium height Δx can be determined.

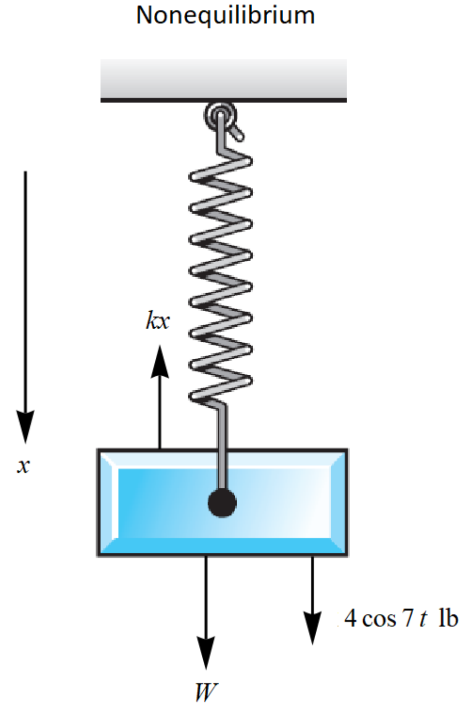
$$6 \text{ lb} = \left(1 \frac{\text{lb}}{\cancel{\text{in}}} \times \frac{12 \cancel{\text{in}}}{1 \text{ ft}}\right) \Delta x$$

$$\Delta x = \frac{1}{2} \text{ ft}$$

The two initial conditions are as follows.

$$x(0) = \frac{1}{2}$$
$$x'(0) = 0$$

Consider now the mass in nonequilibrium.



Apply Newton's second law in the x-direction to obtain the equation of motion for the mass.

$$\sum F_x = ma_x$$

$$-kx + W + 4\cos 7t = ma_x$$

Use the fact that acceleration is the second derivative of position with respect to time.

$$-kx + W + 4\cos 7t = mx''$$

$$mx'' + kx = W + 4\cos 7t$$

This is a linear inhomogeneous ODE, so its general solution can be expressed a sum of the complementary solution and the particular solution.

$$x(t) = x_c(t) + x_p(t)$$

The complementary solution satisfies the associated homogeneous equation.

$$mx_c'' + kx_c = 0 \tag{1}$$

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $x_c = e^{rt}$.

$$x_c = e^{rt} \quad \rightarrow \quad x_c' = re^{rt} \quad \rightarrow \quad x_c'' = r^2 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$m(r^2 e^{rt}) + k(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$mr^2 + k = 0$$
$$r^2 = -\frac{k}{m}$$
$$r = \pm i\sqrt{\frac{k}{m}} = \pm i\omega$$

Two solutions to equation (1) are then

$$x_c = e^{-i\omega t} \quad \text{and} \quad x_c = e^{i\omega t}.$$

By the principle of superposition, the general solution for x_c is a linear combination of these two.

$$\begin{aligned} x_c(t) &= C_1 e^{-i\omega t} + C_2 e^{i\omega t} \\ &= C_1[\cos(-\omega t) + i\sin(-\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1[\cos(\omega t) - i\sin(\omega t)] + C_2[\cos(\omega t) + i\sin(\omega t)] \\ &= C_1\cos\omega t - iC_1\sin\omega t + C_2\cos\omega t + iC_2\sin\omega t \\ &= (C_1 + C_2)\cos\omega t + (-iC_1 + iC_2)\sin\omega t \\ &= C_3\cos\omega t + C_4\sin\omega t \end{aligned}$$

On the other hand, the particular solution satisfies

$$mx_p'' + kx_p = W + 4\cos 7t.$$

For the first term on the right side, we'll include a constant A in the trial solution. Also, since there are only even derivatives on the left side, we'll include $B\cos 3t$ in the trial solution to account for the second term on the right side. Substitute $x_p(t) = A + B\cos 7t$ into the equation to determine A and B.

$$m(A + B\cos 7t)'' + k(A + B\cos 7t) = W + 4\cos 7t$$
$$m(-49B\cos 7t) + kA + kB\cos 7t = W + 4\cos 7t$$
$$kA + (kB - 49mB)\cos 7t = W + 4\cos 7t$$

Match the coefficients.

$$\begin{aligned} kA &= W \\ kB - 49mB &= 4 \end{aligned}$$

Solving this system yields

$$A = \frac{W}{k} \quad \text{and} \quad B = \frac{4}{k - 49m}.$$

The particular solution is then

$$x_p(t) = \frac{W}{k} + \frac{4}{k - 49m}\cos 7t,$$

which means the general solution is

$$\begin{aligned} x(t) &= C_3\cos\omega t + C_4\sin\omega t + \frac{W}{k} + \frac{4}{k - 49m}\cos 7t \\ &= C_3\cos\sqrt{\frac{k}{m}}t + C_4\sin\sqrt{\frac{k}{m}}t + \frac{W}{k} + \frac{4}{k - 49m}\cos 7t. \end{aligned}$$

Take a derivative of it with respect to t.

$$x'(t) = -C_3\sqrt{\frac{k}{m}}\sin\sqrt{\frac{k}{m}}t + C_4\sqrt{\frac{k}{m}}\cos\sqrt{\frac{k}{m}}t - \frac{28}{k-49m}\sin 7t$$

Now apply the initial conditions to determine C_3 and C_4.

$$x(0) = C_3 + \frac{W}{k} + \frac{4}{k-49m} = \frac{1}{2}$$

$$x'(0) = C_4\sqrt{\frac{k}{m}} = 0$$

Solving this system of equations yields

$$C_3 = \frac{1}{2} - \frac{W}{k} - \frac{4}{k-49m} = -\frac{4}{k-49m} \quad \text{and} \quad C_4 = 0,$$

so

$$\begin{aligned} x(t) &= -\frac{4}{k-49m}\cos\sqrt{\frac{k}{m}}t + \frac{W}{k} + \frac{4}{k-49m}\cos 7t \\ &= \frac{W}{k} + \frac{4}{k-49m}\left(\cos 7t - \cos\sqrt{\frac{k}{m}}t\right). \end{aligned}$$

Finally, plug in the numbers, $W = 6$ lb and $k = 12$ lb/ft. The mass m is obtained by dividing the weight by the gravity: $m = W/g = 6\text{ lb}/(32.2\text{ ft/s}^2)$. Note that displacement (a vector) is obtained by subtracting the equilibrium height from $x(t)$.

$$\boldsymbol{d} = x(t) - \Delta x = \frac{4}{k-49m}\left(\cos 7t - \cos\sqrt{\frac{k}{m}}t\right)$$

As $\boldsymbol{d}$ is in feet, multiply the result by 12 to convert it to inches.