Problem 14

In each of Problems 11 through 16, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

$$y^{(4)} + 2y''' + y'' = 0; \qquad 1, \quad t, \quad e^{-t}, \quad te^{-t}$$

Solution

Check that the first solution satisfies the ODE.

$$(1)^{(4)} + 2(1)''' + (1)'' \stackrel{?}{=} 0$$

$$(0) + 2(0) + (0) \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that the second solution satisfies the ODE.

$$(t)^{(4)} + 2(t)''' + (t)'' \stackrel{?}{=} 0$$

$$(0) + 2(0) + (0) \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that the third solution satisfies the ODE.

$$(e^{-t})^{(4)} + 2(e^{-t})''' + (e^{-t})'' \stackrel{?}{=} 0$$

$$(e^{-t}) + 2(-e^{-t}) + (e^{-t}) \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that the fourth solution satisfies the ODE.

$$(te^{-t})^{(4)} + 2(te^{-t})''' + (te^{-t})'' \stackrel{?}{=} 0$$

$$(te^{-t} - 4e^{-t}) + 2(-te^{-t} + 3e^{-t}) + (te^{-t} - 2e^{-t}) \stackrel{?}{=} 0$$

$$0 = 0$$

The Wronskian of the four functions is

$$\begin{aligned} W(1, t, e^{-t}, te^{-t}) &= \begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ (1)' & (t)' & (e^{-t})' & (te^{-t})' \\ (1)'' & (t)'' & (e^{-t})'' & (te^{-t})'' \\ (1)''' & (t)''' & (e^{-t})''' & (te^{-t})''' \end{vmatrix} = \begin{vmatrix} 1 & t & e^{-t} & te^{-t} \\ 0 & 1 & -e^{-t} & e^{-t} - te^{-t} \\ 0 & 0 & e^{-t} & -2e^{-t} + te^{-t} \\ 0 & 0 & -e^{-t} & 3e^{-t} - te^{-t} \end{vmatrix} \\ &= 1\{1[e^{-t}(3e^{-t} - te^{-t}) + e^{-t}(-2e^{-t} + te^{-t})]\} = 3e^{-2t} - te^{-2t} - 2e^{-2t} + te^{-2t} \\ &= e^{-2t}. \end{aligned}$$