Problem 18

Verify that the differential operator defined by

$$L[y] = y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y$$

is a linear differential operator. That is, show that

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2],$$

where y_1 and y_2 are n-times-differentiable functions and c_1 and c_2 are arbitrary constants. Hence, show that if y_1, y_2, ..., y_n are solutions of $L[y] = 0$, then the linear combination $c_1y_1 + \cdots + c_ny_n$ is also a solution of $L[y] = 0$.

Solution

Plug $c_1y_1 + c_2y_2$ for y in the definition of $L[y]$.

$$L[c_1y_1 + c_2y_2] = (c_1y_1 + c_2y_2)^{(n)} + p_1(t)(c_1y_1 + c_2y_2)^{(n-1)} + \cdots + p_n(t)(c_1y_1 + c_2y_2)$$

Use the fact that the derivative of a sum is the sum of the derivatives.

$$L[c_1y_1 + c_2y_2] = (c_1y_1)^{(n)} + (c_2y_2)^{(n)} + p_1(t)[(c_1y_1)^{(n-1)} + (c_2y_2)^{(n-1)}] + \cdots + p_n(t)(c_1y_1 + c_2y_2)$$

Use the fact that constants can be pulled out of the derivative.

$$L[c_1y_1 + c_2y_2] = c_1(y_1)^{(n)} + c_2(y_2)^{(n)} + p_1(t)[c_1(y_1)^{(n-1)} + c_2(y_2)^{(n-1)}] + \cdots + p_n(t)(c_1y_1 + c_2y_2)$$

Use the distributive and commutative properties.

$$L[c_1y_1 + c_2y_2] = c_1y_1^{(n)} + c_2y_2^{(n)} + c_1p_1(t)y_1^{(n-1)} + c_2p_1(t)y_2^{(n-1)} + \cdots + c_1p_n(t)y_1 + c_2p_n(t)y_2$$

Factor c_1 and c_2.

$$L[c_1y_1 + c_2y_2] = c_1\left[y_1^{(n)} + p_1(t)y_1^{(n-1)} + \cdots + p_n(t)y_1\right] + c_2\left[y_2^{(n)} + p_1(t)y_2^{(n-1)} + \cdots + p_n(t)y_2\right]$$

Therefore,

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2].$$

The same argument applies for $c_1y_1 + c_2y_2 + \cdots + c_ny_n$:

$$L[c_1y_1 + c_2y_2 + \cdots + c_ny_n] = c_1L[y_1] + c_2L[y_2] + \cdots + c_nL[y_n]. \tag{1}$$

Suppose that y_1, y_2, ..., y_n are solutions of $L[y] = 0$. Then

$$\begin{array}{cccc} L[y_1] = 0 & L[y_2] = 0 & \cdots & L[y_n] = 0 \\ c_1L[y_1] = 0 & c_2L[y_2] = 0 & \cdots & c_nL[y_n] = 0. \end{array}$$

Add the respective sides of each equation in the bottom row.

$$c_1L[y_1] + c_2L[y_2] + \cdots + c_nL[y_n] = 0$$

Therefore, using equation (1),

$$L[c_1y_1 + c_2y_2 + \cdots + c_ny_n] = 0.$$