Problem 21

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

$$y''' + 2y'' - y' - 3y = 0$$

Solution

Because this is a linear third-order ODE, there will be three solutions for it. Let y_1, y_2, and y_3 represent them and let $W = W(y_1, y_2, y_3)$ be the Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Differentiate both sides with respect to t.

$$W' = \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{=0} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Since y_1, y_2, and y_3 are solutions to the ODE, they satisfy

$$\begin{aligned} y_1''' + 2y_1'' - y_1' - 3y_1 = 0 \quad &\rightarrow \quad y_1''' = -2y_1'' + y_1' + 3y_1 \\ y_2''' + 2y_2'' - y_2' - 3y_2 = 0 \quad &\rightarrow \quad y_2''' = -2y_2'' + y_2' + 3y_2 \\ y_3''' + 2y_3'' - y_3' - 3y_3 = 0 \quad &\rightarrow \quad y_3''' = -2y_3'' + y_3' + 3y_3. \end{aligned}$$

Substitute these formulas for y_1''', y_2''', and y_3''' into the determinant.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -2y_1'' + y_1' + 3y_1 & -2y_2'' + y_2' + 3y_2 & -2y_3'' + y_3' + 3y_3 \end{vmatrix}$$

Multiply the first row by -3, multiply the second row by -1, and add the sum of each column to the respective entry in the third row. Doing so wipes out the latter two terms in each entry.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -2y_1'' & -2y_2'' & -2y_3'' \end{vmatrix} = -2\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -2W$$

Now solve this ODE for W. Divide both sides by W.

$$\frac{W'}{W} = -2$$

The left side can be written as the derivative of a logarithm.

$$\frac{d}{dt}\ln|W| = -2$$

The absolute value sign is included because the logarithm argument cannot be negative. Integrate both sides with respect to t.

$$\ln |W| = -2t + C_1$$

Exponentiate both sides.

$$\begin{aligned} |W| &= e^{-2t+C_1} \\ &= e^{-2t}e^{C_1} \end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$W(t) = \pm e^{C_1}e^{-2t}$$

Use a new constant c for $\pm e^{C_1}$. Therefore,

$$W(t) = ce^{-2t}.$$