# Problem 23

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

$$ty''' + 2y'' - y' + ty = 0$$

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**Solution**

Divide both sides by $t$.

$$y''' + \frac{2}{t}y'' - \frac{1}{t}y' + y = 0$$

Because this is a linear third-order ODE, there will be three solutions for it. Let $y_1$, $y_2$, and $y_3$ represent them and let $W = W(y_1, y_2, y_3)$ be the Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

Differentiate both sides with respect to $t$.

$$W' = \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{=0} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}$$

Since $y_1$, $y_2$, and $y_3$ are solutions to the ODE, they satisfy

$$\begin{aligned} y_1''' + \frac{2}{t}y_1'' - \frac{1}{t}y_1' + y_1 = 0 \quad &\rightarrow \quad y_1''' = -\frac{2}{t}y_1'' + \frac{1}{t}y_1' - y_1 \\ y_2''' + \frac{2}{t}y_2'' - \frac{1}{t}y_2' + y_2 = 0 \quad &\rightarrow \quad y_2''' = -\frac{2}{t}y_2'' + \frac{1}{t}y_2' - y_2 \\ y_3''' + \frac{2}{t}y_3'' - \frac{1}{t}y_3' + y_3 = 0 \quad &\rightarrow \quad y_3''' = -\frac{2}{t}y_3'' + \frac{1}{t}y_3' - y_3. \end{aligned}$$

Substitute these formulas for $y_1'''$, $y_2'''$, and $y_3'''$ into the determinant.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -\frac{2}{t}y_1'' + \frac{1}{t}y_1' - y_1 & -\frac{2}{t}y_2'' + \frac{1}{t}y_2' - y_2 & -\frac{2}{t}y_3'' + \frac{1}{t}y_3' - y_3 \end{vmatrix}$$

Multiply the first row by 1, multiply the second row by $-1/t$, and add the sum of each column to the respective entry in the third row. Doing so wipes out the latter two terms in each entry.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -\frac{2}{t}y_1'' & -\frac{2}{t}y_2'' & -\frac{2}{t}y_3'' \end{vmatrix} = -\frac{2}{t}\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -\frac{2}{t}W$$

Now solve this ODE for $W$. Divide both sides by $W$.

$$\frac{W'}{W} = -\frac{2}{t}$$

The left side can be written as the derivative of a logarithm.

$$\frac{d}{dt}\ln|W| = -\frac{2}{t}$$

The absolute value sign is included because the logarithm argument cannot be negative. Integrate both sides with respect to $t$.

$$\ln|W| = -2\ln t + C_1$$

Exponentiate both sides.

$$\begin{aligned}|W| &= e^{-2\ln t + C_1}\\ &= e^{-2\ln t}e^{C_1}\\ &= e^{\ln t^{-2}}e^{C_1}\\ &= t^{-2}e^{C_1}\end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$W(t) = \pm e^{C_1}t^{-2}$$

Use a new constant $c$ for $\pm e^{C_1}$. Therefore,

$$W(t) = \frac{c}{t^2}.$$