# Problem 24

In each of Problems 21 through 24, use Abel's formula (Problem 20) to find the Wronskian of a fundamental set of solutions of the given differential equation.

$$t^2y^{(4)} + ty''' + y'' - 4y = 0$$

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### Solution

Divide both sides by $t^2$.

$$y^{(4)} + \frac{1}{t}y''' + \frac{1}{t^2}y'' - \frac{4}{t^2}y = 0$$

Because this is a linear fourth-order ODE, there will be four solutions for it. Let $y_1$, $y_2$, $y_3$, and $y_4$ represent them and let $W = W(y_1, y_2, y_3, y_4)$ be the Wronskian.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}$$

Differentiate both sides with respect to $t$.

$$W' = \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} = \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' & y_4' \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1''' & y_2''' & y_3''' & y_4''' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix}}_{=0}$$

$$+ \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1^{(4)} & y_2^{(4)} & y_3^{(4)} & y_4^{(4)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1^{(4)} & y_2^{(4)} & y_3^{(4)} & y_4^{(4)} \end{vmatrix}$$

Since $y_1$, $y_2$, $y_3$, and $y_4$ are solutions to the ODE, they satisfy

$$\begin{aligned} y_1^{(4)} + \frac{1}{t}y_1''' + \frac{1}{t^2}y_1'' - \frac{4}{t^2}y_1 = 0 \quad &\rightarrow \quad y_1^{(4)} = -\frac{1}{t}y_1''' - \frac{1}{t^2}y_1'' + \frac{4}{t^2}y_1 \\ y_2^{(4)} + \frac{1}{t}y_2''' + \frac{1}{t^2}y_2'' - \frac{4}{t^2}y_2 = 0 \quad &\rightarrow \quad y_2^{(4)} = -\frac{1}{t}y_2''' - \frac{1}{t^2}y_2'' + \frac{4}{t^2}y_2 \\ y_3^{(4)} + \frac{1}{t}y_3''' + \frac{1}{t^2}y_3'' - \frac{4}{t^2}y_3 = 0 \quad &\rightarrow \quad y_3^{(4)} = -\frac{1}{t}y_3''' - \frac{1}{t^2}y_3'' + \frac{4}{t^2}y_3 \\ y_4^{(4)} + \frac{1}{t}y_4''' + \frac{1}{t^2}y_4'' - \frac{4}{t^2}y_4 = 0 \quad &\rightarrow \quad y_4^{(4)} = -\frac{1}{t}y_4''' - \frac{1}{t^2}y_4'' + \frac{4}{t^2}y_4. \end{aligned}$$

Substitute these formulas for $y_1'''$, $y_2'''$, $y_3'''$, and $y^{(4)}$ into the determinant.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ -\frac{1}{t}y_1''' - \frac{1}{t^2}y_1'' + \frac{4}{t^2}y_1 & -\frac{1}{t}y_2''' - \frac{1}{t^2}y_2'' + \frac{4}{t^2}y_2 & -\frac{1}{t}y_3''' - \frac{1}{t^2}y_3'' + \frac{4}{t^2}y_3 & -\frac{1}{t}y_4''' - \frac{1}{t^2}y_4'' + \frac{4}{t^2}y_4 \end{vmatrix}$$

Multiply the first row by $-4/t^2$, multiply the third row by $1/t^2$, and add these to the fourth row. Doing so wipes out the latter two terms in each entry.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ -\frac{1}{t}y_1''' & -\frac{1}{t}y_2''' & -\frac{1}{t}y_3''' & -\frac{1}{t}y_4''' \end{vmatrix} = -\frac{1}{t}\begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} = -\frac{1}{t}W$$

Now solve this ODE for $W$. Divide both sides by $W$.

$$\frac{W'}{W} = -\frac{1}{t}$$

The left side can be written as the derivative of a logarithm.

$$\frac{d}{dt}\ln|W| = -\frac{1}{t}$$

The absolute value sign is included because the logarithm argument cannot be negative. Integrate both sides with respect to $t$.

$$\ln|W| = -\ln t + C_1$$

Exponentiate both sides.

$$\begin{aligned} |W| &= e^{-\ln t + C_1} \\ &= e^{-\ln t}e^{C_1} \\ &= e^{\ln t^{-1}}e^{C_1} \\ &= t^{-1}e^{C_1} \end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$W(t) = \pm e^{C_1}t^{-1}$$

Use a new constant $c$ for $\pm e^{C_1}$. Therefore,

$$W(t) = \frac{c}{t}.$$