Problem 27

In each of Problems 27 and 28, use the method of reduction of order (Problem 26) to solve the given differential equation.

$$(2-t)y''' + (2t-3)y'' - ty' + y = 0, \quad t < 2; \qquad y_1(t) = e^t$$

Solution

According to the method of reduction of order, the general solution can be obtained by plugging in $y(t) = c(t)e^t$ to the ODE.

$$(2-t)[c(t)e^t]''' + (2t-3)[c(t)e^t]'' - t[c(t)e^t]' + [c(t)e^t] = 0$$

Evaluate the derivatives.

$$(2-t)[c'(t)e^t + c(t)e^t]'' + (2t-3)[c'(t)e^t + c(t)e^t]' - t[c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0$$

$$(2-t)[c''(t)e^t + 2c'(t)e^t + c(t)e^t]' + (2t-3)[c''(t)e^t + 2c'(t)e^t + c(t)e^t] - t[c'(t)e^t + c(t)e^t] + [c(t)e^t] = 0$$

$$\begin{aligned}(2-t)[c'''(t)e^t + 3c''(t)e^t + 3c'(t)e^t + c(t)e^t]& \\ + (2t-3)[c''(t)e^t + 2c'(t)e^t + c(t)e^t] - t[c'(t)e^t + c(t)e^t] + [c(t)e^t] &= 0\end{aligned}$$

Expand the left side.

$$\begin{aligned}2c'''(t)e^t + 6c''(t)e^t + 6c'(t)e^t + \cancel{2c(t)e^t} - tc'''(t)e^t - 3tc''(t)e^t - \cancel{3tc'(t)e^t} - \cancel{tc(t)e^t}& \\ + 2tc''(t)e^t + \cancel{4tc'(t)e^t} + \cancel{2tc(t)e^t} - 3c''(t)e^t - 6c'(t)e^t - \cancel{3c(t)e^t}& \\ - \cancel{tc'(t)e^t} - \cancel{tc(t)e^t} + \cancel{c(t)e^t} &= 0\end{aligned}$$

$$(2-t)c'''(t)e^t + (3-t)c''(t)e^t + \cancel{6c'(t)e^t} - \cancel{6c'(t)e^t} = 0$$

$$(2-t)c'''(t)e^t + (3-t)c''(t)e^t = 0$$

Divide both sides by $(2-t)e^t$.

$$c'''(t) + \frac{3-t}{2-t}c''(t) = 0$$

Bring the second term to the right side and divide both sides by $c''(t)$.

$$\frac{c'''(t)}{c''(t)} = -\frac{3-t}{2-t}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln|c''(t)| = \frac{t-3}{2-t}$$

An absolute value sign has been included because the logarithm argument cannot be negative. Integrate both sides with respect to t.

$$\ln|c''(t)| = \ln|t-2| - t + C_1$$

Exponentiate both sides.

$$\begin{aligned}|c''(t)| &= e^{\ln|t-2|-t+C_1} \\ &= e^{\ln|t-2|}e^{-t}e^{C_1} \\ &= |t-2|e^{-t}e^{C_1}\end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$c''(t) = \pm e^{C_1}|t-2|e^{-t}$$

Use a new constant C_2 for $\pm e^{C_1}$. Also, since $t < 2$, the remaining absolute value sign can be removed by using $(2-t)$.

$$c''(t) = C_2(2-t)e^{-t}$$

Integrate both sides with respect to t again.

$$c'(t) = C_2(t-1)e^{-t} + C_3$$

Integrate both sides with respect to t once more.

$$c(t) = -C_2te^{-t} + C_3t + C_4$$

Therefore, since $y(t) = c(t)e^t$, the general solution is

$$y(t) = C_5t + C_3te^t + C_4e^t,$$

where a new constant C_5 was used for $-C_2$.