Problem 28

In each of Problems 27 and 28, use the method of reduction of order (Problem 26) to solve the given differential equation.

$$t^2(t+3)y''' - 3t(t+2)y'' + 6(1+t)y' - 6y = 0, \quad t > 0; \qquad y_1(t) = t^2, \quad y_2(t) = t^3$$

Solution

According to the method of reduction of order, the general solution can be obtained by plugging in $y(t) = c(t)t^2$ to the ODE.

$$t^2(t+3)[c(t)t^2]''' - 3t(t+2)[c(t)t^2]'' + 6(1+t)[c(t)t^2]' - 6[c(t)t^2] = 0$$

Evaluate the derivatives.

$$t^2(t+3)[c'(t)t^2 + 2tc(t)]'' - 3t(t+2)[c'(t)t^2 + 2tc(t)]' + 6(1+t)[c'(t)t^2 + 2tc(t)] - 6[c(t)t^2] = 0$$

$$\begin{aligned} t^2(t+3)[c''(t)t^2 + 4tc'(t) + 2c(t)]' - 3t(t+2)[c''(t)t^2 + 4tc'(t) + 2c(t)] \\ + 6(1+t)[c'(t)t^2 + 2tc(t)] - 6[c(t)t^2] = 0 \end{aligned}$$

$$\begin{aligned} t^2(t+3)[c'''(t)t^2 + 6tc''(t) + 6c'(t)] - 3t(t+2)[c''(t)t^2 + 4tc'(t) + 2c(t)] \\ + 6(1+t)[c'(t)t^2 + 2tc(t)] - 6[c(t)t^2] = 0 \end{aligned}$$

Simplify the left side.

$$t^4(t+3)c'''(t) + 3t^3(t+4)c''(t) = 0$$

Divide both sides by $t^4(t+3)$.

$$c'''(t) + \frac{3(t+4)}{t(t+3)}c''(t) = 0$$

Bring the second term to the right side and divide both sides by $c''(t)$.

$$\frac{c'''(t)}{c''(t)} = -\frac{3(t+4)}{t(t+3)}$$

The left side can be written as the derivative of a logarithm by the chain rule.

$$\frac{d}{dt}\ln|c''(t)| = -\frac{3t+12}{t(t+3)}$$

An absolute value sign has been included because the logarithm argument cannot be negative. Integrate both sides with respect to t.

$$\begin{aligned} \ln|c''(t)| &= \int^t -\frac{3s+12}{s(s+3)}\,ds \\ &= \int^t \left(\frac{1}{s+3} - \frac{4}{s}\right) ds \\ &= \ln|t+3| - 4\ln|t| + C_1 \end{aligned}$$

Since $t > 0$, the absolute value signs on the right side can be dropped. Exponentiate both sides.

$$\begin{aligned} |c''(t)| &= e^{\ln(t+3) - 4\ln t + C_1} \\ &= e^{\ln(t+3)} e^{-4\ln t} e^{C_1} \\ &= e^{\ln(t+3)} e^{\ln t^{-4}} e^{C_1} \\ &= (t+3)t^{-4} e^{C_1} \end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$c''(t) = \pm e^{C_1}(t+3)t^{-4}$$

Use a new constant C_2 for $\pm e^{C_1}$.

$$c''(t) = C_2(t^{-3} + 3t^{-4})$$

Integrate both sides with respect to t again.

$$c'(t) = C_2\left(-\frac{1}{2}t^{-2} - t^{-3}\right) + C_3$$

Integrate both sides with respect to t once more.

$$\begin{aligned} c(t) &= C_2\left(\frac{1}{2}t^{-1} + \frac{1}{2}t^{-2}\right) + C_3 t + C_4 \\ &= \frac{C_2}{2}(t^{-1} + t^{-2}) + C_3 t + C_4 \end{aligned}$$

Therefore, since $y(t) = c(t)t^2$, the general solution is

$$y(t) = C_5(t+1) + C_3 t^3 + C_4 t^2,$$

where a new constant C_5 was used for $C_2/2$.