Problem 13

In each of Problems 11 through 16, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

$$y''' + 2y'' - y' - 2y = 0; \qquad e^t, \quad e^{-t}, \quad e^{-2t}$$

Solution

Check that the first solution satisfies the ODE.

$$(e^t)''' + 2(e^t)'' - (e^t)' - 2(e^t) \stackrel{?}{=} 0$$

$$(e^t) + 2(e^t) - (e^t) - 2(e^t) \stackrel{?}{=} 0$$
$$0 = 0$$

Now check that the second solution satisfies the ODE.

$$(e^{-t})''' + 2(e^{-t})'' - (e^{-t})' - 2(e^{-t}) \stackrel{?}{=} 0$$

$$(-e^{-t}) + 2(e^{-t}) - (-e^{-t}) - 2(e^{-t}) \stackrel{?}{=} 0$$
$$0 = 0$$

Now check that the third solution satisfies the ODE.

$$(e^{-2t})''' + 2(e^{-2t})'' - (e^{-2t})' - 2(e^{-2t}) \stackrel{?}{=} 0$$

$$(-8e^{-2t}) + 2(4e^{-2t}) - (-2e^{-2t}) - 2(e^{-2t}) \stackrel{?}{=} 0$$
$$0 = 0$$

The Wronskian of the three functions is

$$\begin{aligned} W(e^t, e^{-t}, e^{-2t}) &= \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ (e^t)' & (e^{-t})' & (e^{-2t})' \\ (e^t)'' & (e^{-t})'' & (e^{-2t})'' \end{vmatrix} \\ &= \begin{vmatrix} e^t & e^{-t} & e^{-2t} \\ e^t & -e^{-t} & -2e^{-2t} \\ e^t & e^{-t} & 4e^{-2t} \end{vmatrix} \\ &= e^t(-4e^{-3t} + 2e^{-3t}) - e^t(4e^{-3t} - e^{-3t}) + e^t(-2e^{-3t} + e^{-3t}) \\ &= -6e^{-2t}. \end{aligned}$$