Problem 16

In each of Problems 11 through 16, verify that the given functions are solutions of the differential equation, and determine their Wronskian.

$$x^3y''' + x^2y'' - 2xy' + 2y = 0; \qquad x, \quad x^2, \quad 1/x$$

Solution

Check that the first solution satisfies the ODE.

$$x^3(x)''' + x^2(x)'' - 2x(x)' + 2(x) \stackrel{?}{=} 0$$

$$x^3(0) + x^2(0) - 2x(1) + 2(x) \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that the second solution satisfies the ODE.

$$x^3(x^2)''' + x^2(x^2)'' - 2x(x^2)' + 2(x^2) \stackrel{?}{=} 0$$

$$x^3(0) + x^2(2) - 2x(2x) + 2(x^2) \stackrel{?}{=} 0$$

$$0 = 0$$

Now check that the third solution satisfies the ODE.

$$x^3(1/x)''' + x^2(1/x)'' - 2x(1/x)' + 2(1/x) \stackrel{?}{=} 0$$

$$x^3(-6/x^4) + x^2(2/x^3) - 2x(-1/x^2) + 2(1/x) \stackrel{?}{=} 0$$

$$0 = 0$$

The Wronskian of the three functions is

$$\begin{aligned} W(x, x^2, 1/x) &= \begin{vmatrix} x & x^2 & 1/x \\ (x)' & (x^2)' & (1/x)' \\ (x)'' & (x^2)'' & (1/x)'' \end{vmatrix} \\ &= \begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} \\ &= \frac{2}{x^3}(2x^2 - x^2) - 2(-1/x - 1/x) \\ &= \frac{2}{x} + \frac{4}{x} \\ &= \frac{6}{x}. \end{aligned}$$