Problem 17

Show that $W(5, \sin^2 t, \cos 2t) = 0$ for all t. Can you establish this result without direct evaluation of the Wronskian?

Solution

Evaluate the Wronskian.

$$\begin{aligned}
W(5, \sin^2 t, \cos 2t) &= \begin{vmatrix} 5 & \sin^2 t & \cos 2t \\ (5)' & (\sin^2 t)' & (\cos 2t)' \\ (5)'' & (\sin^2 t)'' & (\cos 2t)'' \end{vmatrix} \\
&= \begin{vmatrix} 5 & \sin^2 t & \cos 2t \\ 0 & 2\sin t \cos t & -2\sin 2t \\ 0 & 2\cos^2 t - 2\sin^2 t & -4\cos 2t \end{vmatrix} \\
&= 5 \begin{vmatrix} 2\sin t \cos t & -2\sin 2t \\ 2\cos^2 t - 2\sin^2 t & -4\cos 2t \end{vmatrix} \\
&= 5[(2\sin t \cos t(-4\cos 2t) + 2\sin 2t(2\cos^2 t - 2\sin^2 t)] \\
&= 5[\sin 2t(-4\cos 2t) + 4\sin 2t(\cos^2 t - \sin^2 t)] \\
&= 5(-4\sin 2t \cos 2t + 4\sin 2t \cos 2t) \\
&= 5(0) \\
&= 0
\end{aligned}$$

This result could have been established without calculating the Wronskian by recognizing that $\cos 2t$ could be written as a linear combination of 5 and $\sin^2 t$.

$$\begin{aligned}
\cos 2t &= 1 - 2\sin^2 t \\
&= \frac{1}{5}(5) - 2(\sin^2 t)
\end{aligned}$$