Problem 19

Let the linear differential operator L be defined by

$$L[y] = a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y,$$

where a_0, a_1, ..., a_n are real constants.

(a) Find $L[t^n]$.

(b) Find $L[e^{rt}]$.

(c) Determine four solutions of the equation $y^{(4)} - 5y'' + 4y = 0$. Do you think the four solutions form a fundamental set of solutions? Why?

Solution

Part (a)

$$\begin{aligned}
L[t^n] &= a_0 (t^n)^{(n)} + a_1 (t^n)^{(n-1)} + \cdots + a_n (t^n) \\
&= a_0 \frac{d^n}{dt^n}(t^n) + a_1 \frac{d^{n-1}}{dt^{n-1}}(t^n) + \cdots + a_n t^n \\
&= a_0 n! t^{n-n} + a_1 \frac{n!}{1} t^{n-(n-1)} + a_2 \frac{n!}{1 \cdot 2} t^{n-(n-2)} + \cdots + a_n t^n \\
&= \sum_{k=0}^{n} a_k \frac{n!}{k!} t^k
\end{aligned}$$

Part (b)

$$\begin{aligned}
L[e^{rt}] &= a_0 (e^{rt})^{(n)} + a_1 (e^{rt})^{(n-1)} + \cdots + a_n (e^{rt}) \\
&= a_0 \frac{d^n}{dt^n}(e^{rt}) + a_1 \frac{d^{n-1}}{dt^{n-1}}(e^{rt}) + \cdots + a_n e^{rt} \\
&= a_0 r^n e^{rt} + a_1 r^{n-1} e^{rt} + \cdots + a_n e^{rt} \\
&= e^{rt}(a_0 r^n + a_1 r^{n-1} + \cdots + a_n) \\
&= e^{rt} \sum_{k=0}^{n} a_k r^{n-k}
\end{aligned}$$

Part (c)

Since the coefficients are constant and this is a homogeneous ODE, the solutions are of the form $y = e^{rt}$.

$$y = e^{rt} \quad \rightarrow \quad y' = re^{rt} \quad \rightarrow \quad y'' = r^2 e^{rt} \quad \rightarrow \quad y''' = r^3 e^{rt} \quad \rightarrow \quad y^{(4)} = r^4 e^{rt}$$

Substitute these expressions to obtain an algebraic equation for r.

$$r^4 e^{rt} - 5(r^2 e^{rt}) + 4(e^{rt}) = 0$$

Divide both sides by e^{rt}.

$$r^4 - 5r^2 + 4 = 0$$

$$(r^2 - 4)(r^2 - 1) = 0$$

$$r = \{-2, -1, 1, 2\}$$

Four solutions to equation (1) are $y = e^{-2t}$ and $y = e^{-t}$ and $y = e^t$ and $y = e^{2t}$. These four solutions form a fundamental set of solutions because the Wronskian is not zero:

$$W(e^{-2t}, e^{-t}, e^t, e^{2t}) = \begin{vmatrix} e^{-2t} & e^{-t} & e^t & e^{2t} \\ (e^{-2t})' & (e^{-t})' & (e^t)' & (e^{2t})' \\ (e^{-2t})'' & (e^{-t})'' & (e^t)'' & (e^{2t})'' \\ (e^{-2t})''' & (e^{-t})''' & (e^t)''' & (e^{2t})''' \end{vmatrix} = \begin{vmatrix} e^{-2t} & e^{-t} & e^t & e^{2t} \\ -2e^{-2t} & -e^{-t} & e^t & 2e^{2t} \\ 4e^{-2t} & e^{-t} & e^t & 4e^{2t} \\ -8e^{-2t} & -e^{-t} & e^t & 8e^{2t} \end{vmatrix} = 72.$$