# Problem 20

In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation

$$y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0.$$

Let $y_1$, $y_2$, and $y_3$ be solutions of this equation on an interval $I$.

(a) If $W = W(y_1, y_2, y_3)$, show that

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

*Hint:* The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

(b) Substitute for $y_1'''$, $y_2'''$, and $y_3'''$ from the differential equation; multiply the first row by $p_3$, multiply the second row by $p_2$, and add these to the last row to obtain

$$W' = -p_1(t)W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c \exp\left[-\int p_1(t)\,dt\right].$$

It follows that $W$ is either always zero or nowhere zero on $I$.

(d) Generalize this argument to the $n$th order equation

$$y^{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0$$

with solutions $y_1, \ldots, y_n$. That is, establish Abel's formula

$$W(y_1, \ldots, y_n)(t) = c \exp\left[-\int p_1(t)\,dt\right]$$

for this case.

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**Solution**

## Part (a)

Let $W = W(y_1, y_2, y_3)$ be the Wronskian of $y_1$, $y_2$, and $y_3$.

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \\ &= y_1 \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix} \\ &= y_1(y_2'y_3'' - y_2''y_3') - y_2(y_1'y_3'' - y_1''y_3') + y_3(y_1'y_2'' - y_1''y_2') \end{aligned}$$

Differentiate both sides with respect to $t$.

$$\begin{aligned} W' = y_1'(y_2'y_3'' - y_2''y_3') + y_1(\cancel{y_2''y_3''} + y_2'y_3''' - y_2'''y_3' - \cancel{y_2''y_3''}) - y_2'(y_1'y_3'' - y_1''y_3') \\ - y_2(\cancel{y_1''y_3''} + y_1'y_3''' - y_1'''y_3' - \cancel{y_1''y_3''}) + y_3'(y_1'y_2'' - y_1''y_2') \\ + y_3(\cancel{y_1''y_2''} + y_1'y_2''' - y_1'''y_2' - \cancel{y_1''y_2''}) \end{aligned}$$

Expand the right side.

$$\begin{aligned} W' = \cancel{y_1'y_2'y_3''} - \cancel{y_1'y_2''y_3'} + y_1y_2'y_3''' - y_1y_2'''y_3' - \cancel{y_1'y_2'y_3''} + \cancel{y_1''y_2'y_3'} \\ - y_1'y_2y_3''' + y_1'''y_2y_3' + \cancel{y_1'y_2''y_3'} - \cancel{y_1''y_2'y_3'} + y_1'y_2'''y_3 - y_1'''y_2'y_3 \end{aligned}$$

Now factor $y_1$, $y_2$, and $y_3$ from the remaining terms.

$$W' = y_1(y_2'y_3''' - y_2'''y_3') - y_2(y_1'y_3''' - y_1'''y_3') + y_3(y_1'y_2''' - y_1'''y_2') \tag{1}$$

The quantities in parentheses can be expressed as determinants.

$$W' = y_1 \begin{vmatrix} y_2' & y_3' \\ y_2''' & y_3''' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1''' & y_3''' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1''' & y_2''' \end{vmatrix}$$

Therefore,

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Alternatively, using the hint,

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \quad \rightarrow \quad W' = \underbrace{\begin{vmatrix} y_1' & y_2' & y_3' \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}}_{=0} + \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

## Part (b)

Since $y_1$, $y_2$, and $y_3$ are solutions to the ODE, they satisfy

$$\begin{aligned} y_1''' + p_1(t)y_1'' + p_2(t)y_1' + p_3(t)y_1 = 0 \quad &\rightarrow \quad y_1''' = -p_1(t)y_1'' - p_2(t)y_1' - p_3(t)y_1 \\ y_2''' + p_1(t)y_2'' + p_2(t)y_2' + p_3(t)y_2 = 0 \quad &\rightarrow \quad y_2''' = -p_1(t)y_2'' - p_2(t)y_2' - p_3(t)y_2 \\ y_3''' + p_1(t)y_3'' + p_2(t)y_3' + p_3(t)y_3 = 0 \quad &\rightarrow \quad y_3''' = -p_1(t)y_3'' - p_2(t)y_3' - p_3(t)y_3 \end{aligned}$$

Plug these formulas for $y_1'''$, $y_2'''$, and $y_3'''$ into equation (1).

$$\begin{aligned} W' &= y_1\{y_2'[-p_1(t)y_3'' - p_2(t)y_3' - p_3(t)y_3] - [-p_1(t)y_2'' - p_2(t)y_2' - p_3(t)y_2]y_3'\} \\ &\quad - y_2\{y_1'[-p_1(t)y_3'' - p_2(t)y_3' - p_3(t)y_3] - [-p_1(t)y_1'' - p_2(t)y_1' - p_3(t)y_1]y_3'\} \\ &\quad + y_3\{y_1'[-p_1(t)y_2'' - p_2(t)y_2' - p_3(t)y_2] - [-p_1(t)y_1'' - p_2(t)y_1' - p_3(t)y_1]y_2'\} \\ &= y_1[-p_1(t)y_2'y_3'' + p_1(t)y_2''y_3' - \cancel{p_3(t)y_2'y_3} + \cancel{p_3(t)y_2y_3'}] \\ &\quad - y_2[-p_1(t)y_1'y_3'' + p_1(t)y_1''y_3' - \cancel{p_3(t)y_1'y_3} + \cancel{p_3(t)y_1y_3'}] \\ &\quad + y_3[-p_1(t)y_1'y_2'' + p_1(t)y_1''y_2' - \cancel{p_3(t)y_1'y_2} + \cancel{p_3(t)y_1y_2'}] \\ &= -p_1(t)[y_1(y_2'y_3'' - y_2''y_3') - y_2(y_1'y_3'' - y_1''y_3') + y_3(y_1'y_2'' - y_1''y_2')] \\ &= -p_1(t)W \end{aligned}$$

Alternatively, we can proceed as we're told.

$$\begin{aligned} W' &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix} \\ &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1(t)y_1'' - p_2(t)y_1' - p_3(t)y_1 & -p_1(t)y_2'' - p_2(t)y_2' - p_3(t)y_2 & -p_1(t)y_3'' - p_2(t)y_3' - p_3(t)y_3 \end{vmatrix} \end{aligned}$$

Multiply the first row by $p_3(t)$, multiply the second row by $p_2(t)$, and add the sum of each column to the respective entry in the third row. Doing so wipes out the latter two terms in each entry.

$$W' = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ -p_1(t)y_1'' & -p_1(t)y_2'' & -p_1(t)y_3'' \end{vmatrix} = -p_1(t) \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = -p_1(t)W$$

**Part (c)**

Now solve this ODE for $W$. Divide both sides by $W$.

$$\frac{W'}{W} = -p_1(t)$$

The left side can be written as the derivative of a logarithm.

$$\frac{d}{dt} \ln |W| = -p_1(t)$$

The absolute value sign is included because the logarithm argument cannot be negative. Integrate both sides with respect to $t$.

$$\begin{aligned} \ln |W| &= \int^t [-p_1(s)]\, ds + C_1 \\ &= -\int^t p_1(s)\, ds + C_1 \end{aligned}$$

Exponentiate both sides.

$$\begin{aligned} |W| &= \exp\left[-\int^t p_1(s)\, ds + C_1\right] \\ &= \exp\left[-\int^t p_1(s)\, ds\right] e^{C_1} \end{aligned}$$

Place $\pm$ on the right side to remove the absolute value sign on the left.

$$W(t) = \pm e^{C_1} \exp\left[-\int^t p_1(s)\, ds\right]$$

Use a new constant $c$ for $\pm e^{C_1}$. Therefore,

$$W(t) = c \exp\left[-\int^t p_1(s)\, ds\right].$$

## Part (d)

Let $W = W(y_1, y_2, \dots, y_n)$ be the Wronskian of $y_1$, $y_2$, $\dots$, and $y_n$.

$$W = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

Differentiate both sides with respect to $t$.

$$W' = \underbrace{\begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}}_{=0} + \underbrace{\begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}}_{=0} + \cdots + \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

$$= \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

Since $y_1$, $y_2$, $\dots$, and $y_n$ are solutions to the ODE, they satisfy

$$\begin{aligned} y_1^{(n)} + p_1(t)y_1^{(n-1)} + \cdots + p_n(t)y_1 = 0 \quad &\rightarrow \quad y_1^{(n)} = -p_1(t)y_1^{(n-1)} - \cdots - p_n(t)y_1 \\ y_2^{(n)} + p_1(t)y_2^{(n-1)} + \cdots + p_n(t)y_2 = 0 \quad &\rightarrow \quad y_2^{(n)} = -p_1(t)y_2^{(n-1)} - \cdots - p_n(t)y_2 \\ &\vdots \\ y_n^{(n)} + p_1(t)y_n^{(n-1)} + \cdots + p_n(t)y_n = 0 \quad &\rightarrow \quad y_n^{(n)} = -p_1(t)y_n^{(n-1)} - \cdots - p_n(t)y_n. \end{aligned}$$

Substitute these formulas for $y_1^{(n)}$, $y_2^{(n)}$,$\dots$, and $y_n^{(n)}$ into the determinant.

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ -p_1(t)y_1^{(n-1)} - \cdots - p_n(t)y_1 & -p_1(t)y_2^{(n-1)} - \cdots - p_n(t)y_2 & \cdots & -p_1(t)y_n^{(n-1)} - \cdots - p_n(t)y_n \end{vmatrix}$$

Multiply the first row by $p_n(t)$, multiply the second row by $p_{n-1}(t)$, and so on. Then add the sum of each column to the respective entry in the third row. Doing so wipes out the latter $n-1$ terms in each entry.

$$W' = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ -p_1(t)y_1^{(n-1)} & -p_1(t)y_2^{(n-1)} & \cdots & -p_1(t)y_n^{(n-1)} \end{vmatrix} = -p_1(t) \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} = -p_1(t)W$$